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Split string field theory II

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Abstract: We describe the ghost sector of cubic string field theory in terms of degrees of freedom on the two halves of a split string. In particular, we represent a class of pure ghost BRST operators as operators on the space of half-string functionals. These BRST operators were postulated by Rastelli, Sen, and Zwiebach to give a description of cubic string field theory in the closed string vacuum arising from condensation of a D25-brane in the original tachyonic theory. We find a class of solutions for the ghost equations of motion using the pure ghost BRST operators. We find a vanishing action for these solutions, and discuss possible interpretations of this result. The form of the solutions we find in the pure ghost theory suggests an analogous class of solutions in the original theory on the D25-brane with BRST operator Q_B coupling the matter and ghost sectors.

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Contents

1. Introduction

From the beginning string theory was a theory of open strings. It began as a model for an S-matrix [1], later interpreted as the S-matrix of open strings. Closed strings were discovered as singularities in the one-loop annulus open string scattering diagram [2], and interpreted as new states in the critical dimension (26 or 10) [3]. This was the first manifestation of one of the new and most important features of string theory — namely what is now called the UV-IR connection. Closed string states emerged not as dynamical bound states of open strings, but rather appeared in closed string channels, to lowest order in the string coupling, arising from the ultraviolet behavior of open string propagators.

Once it became clear that string theory is a theory of quantum gravity, and has many possible consistent classical backgrounds, it became imperative to find a background-independent formulation of the theory. A natural strategy was to construct a *string field theory*, in which solutions of the classical string equations of motion would correspond to different backgrounds, much as solutions of Einstein's equations yield different background geometries in general relativity. Witten's open string field theory raised hopes in this direction, yielding a off-shell Chern-Simons-like theory of string functionals [4], that reproduces open string scattering amplitudes [5]. This theory was generalized to open superstrings [6, 7] and to a theory of open and closed strings [8], although the formulation of closed string field theory is much more complicated. The rather complicated form of the explicit realization of string field theory, however, made it difficult to treat the theory, even at the classical level.

Because open string scattering amplitudes contain closed string poles, the possibility has always existed that a complete formulation of string theory might be given in terms of open strings alone. This possibility has been greatly supported by recent developments. First, the web of dualities has revealed the unity of all previous formulations of string theory. Closed string theories are connected, through such dualities, to open string theories. Second, the discovery of D-branes has revealed the existence of new dynamical objects in closed string theory. These D-branes can be thought of as solitons of open string theory, whose tension behaves characteristically as $1/g_{\rm open}^2 = 1/g_{\rm closed}$, and whose whose dynamics is controlled by open strings. Finally, AdS/CFT duality provides a rich class of examples whereby open string theory captures the full essence of closed string theory in a particular background [9, 10, 11]. The most remarkable feature of this duality is that SUSY Yang Mills theory in four dimensions, the low-energy limit of the open string theory on a D3-brane, is rich enough to capture the physics of closed string theory in the bulk of $\mathrm{AdS}_5 \times S^5$. The UV-IR connection plays an essential role in this scenario.

The example of AdS/CFT duality once again raises the possibility of giving a nonperturbative definition of string theory in terms of open string theory, indeed, in this case, in terms of the low energy limit of this theory — ordinary (supersymmetric) Yang-Mills field theory. However, background independence seems out of reach in this framework. To alter the spacetime background in the bulk requires the insertion of operators of dimension greater than four in the gauge theory. It is not known what regularization process is required or how to handle such operators. Once again, to control the infra-red in the closed string sector we need full control over the ultra-violet in the open string sector. Furthermore, it is clear that to capture the full moduli space of closed string backgrounds one would have to consider extensions of gauge theory with an infinite number of non-renormalizable interactions. Even a simple modification of the background, such as turning on a background $B_{\mu\nu}$ field, requires a major modification of the theory, namely a deformation that takes one to a non-commutative field theory. Non-commutative field theory is the one known example of a controllable deformation of gauge theory that involves an infinite number of non-renormalizable interactions. Standard field theoretic methods are unable to deal with the general case. String theory, however, knows how to control the ultra-violet, so that the prospect remains that open string field theory could capture the full essence of closed string theory, including the full moduli space of closed string vacua.

Open string field theory has recently come back into full focus, due largely to the remarkable conjectures of Ashoke Sen, who suggested that the tachyon of the ordinary 26-dimensional bosonic string theory should be thought of as an instability of the space-filling D25-branes on which this theory is constructed [12]. He argued that the condensation of these tachyons defines a (classically) stable vacuum of the theory in which the D-branes have annihilated, and that about this vacuum configuration there should be no finite mass open string excitations — this vacuum should instead describe the 26-dimensional closed string theory [13]. Much evidence in support of these conjectures has been given, using methods of level truncation and using boundary string field theory (see paper I [14] for a list of references, and [15] for a review of this work). If true, these conjectures strongly support the notion that open string theory is powerful enough to contain classical D-brane solutions, and perhaps the full content of string theory.

To realize these goals much remains to be done. First, one needs to achieve analytic mastery of the classical string field theory equations of motion and to construct analytic D-brane solutions. Second, one needs to show that at the stable vacuum there are no open string physical excitations and that at the one-loop level the ordinary closed string spectrum of physical states emerge. Third, in order to achieve full background independence one needs to cast the formalism of open string field theory into an abstract framework, one that does not require a particular closed string background, or a particular conformal field theory, for its formulation. In such a formalism the closed string background would emerge as a consequence of solving the equations of motion.

To this end, we attempt to give an operator formulation of string field theory in which the string fields are defined as operators in an appropriate space. Roughly speaking the space is the space of half-string states, and string fields, which are normally thought of as functions of open strings $\Psi[x(\sigma)]$, $0 \leq \sigma \leq \pi$ are instead regarded as matrices $\Psi_{l,r}$, where $l(\sigma) = x(\sigma)$ and $r(\sigma) = x(\pi - \sigma)$ for $0 \leq \sigma \leq \pi/2$. With such an identification Witten's $\star$ product is roughly described by matrix multiplication.

Although such an interpretation was suggested in the early papers on open string field theory [4, 16, 17], it appeared impossible to precisely realize such a formalism because of difficulties associated with the midpoint of the string. We are attempting to confront these difficulties, starting with the very concrete problem of analytically verifying Sen's conjectures, with the hope of then generalizing the algebraic framework in a way that could be truly background independent. Other approaches to a half-string operator formalism for open string field theory have appeared in [18]–[23].

In our previous paper [14] (henceforth (I)), we approached these issues in a simplified setting, following the suggestion of Rastelli, Sen and Zwiebach [24] that in the locally stable vacuum in which the space-filling D25-brane has been annihilated, it may be possible to describe the shifted string field theory using a BRST operator Q that is pure ghost. If true, this explains the decoupling of open strings at the stable vacuum; since the cohomology of such a Q is trivial, there are simply no perturbative physical states. There can, of course, be physical non-perturbative solitons, these should be the D-branes of the closed string theory.

We would expect, even though there are no physical states in the classical open string field theory around the closed string vacuum, that closed string states should appear at one loop in off-shell open string correlation functions. An interesting analogy to this situation is QCD, in which there are no physical states corresponding to the quark and gluon fields in the action, yet hadrons appear as physical states in quark and gluon correlation functions. The situation here is somewhat different. Quarks and gluons disappear from the physical spectrum (are confined) only in a non-perturbative treatment of the quantum theory — here the decoupling occurs at the classical level. Hadrons only appear as non-perturbative bound states of quarks and gluons — here closed strings should appear at one loop due to the UV-IR connection. Another, perhaps useful analogy, is QCD_2, in which, depending on the gauge choice or regularization of the gluon propagator, quarks are either (i) infinitely massive and decouple from the spectrum even off-shell (the *singular* gauge); or (ii) have a finite mass but decouple on-shell from physical color singlet hadrons (the *regular* gauge). The RSZ description of the vacuum is analogous to the singular gauge where the decoupling of quarks is trivial. There might very well be a less singular description, in which off-shell open string states exist with finite energy, but such that these states decouple on shell. In such a description it might be much easier to find the closed string states, since it is difficult (requires some kind of limiting procedure) to obtain closed string states from the ultraviolet behavior of the open string propagator when this propagator is singular. Numerical evidence [25, 26] suggests that all open string degrees of freedom decouple in the locally stable vacuum in Witten's original description of the theory with BRST operator Q_B. If we can find an analytic description of this vacuum, this description of the theory may turn out to be the best framework in which to investigate the behavior of closed string degrees of freedom.

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With a pure ghost Q, of the kind suggested by RSZ, the string field equations of motion, $Q\Psi + \Psi \star \Psi = 0$, can be factorized into separate ghost and matter equations, when the full string field takes the form $\Psi = \Psi_m \otimes \Psi_g$. In the matter sector the relevant equation is simply the projection equation, $\Psi_m = \Psi_m \star \Psi_m$. This factorization was used in [27] to identify certain solutions of the matter equation, originally found in [28, 29], as the matter part of a D25-brane solution. In (I) we developed a split-string formalism for the matter sector of the theory, and showed that these solutions of the matter equation are indeed projection operators onto certain functionals on the space of half-string configurations; related work appeared in [22]. A rank N projection operator corresponds to a system of N D-branes in the RSZ model. For example, a simple class of rank one projection operators take the form $e^{-l\cdot M\cdot l - r\cdot M\cdot r}$, where l, r describe the degrees of freedom of the left and right halves of the string. We showed that the sliver state discussed in [28, 29, 27] is a projection operator of this form, and that for functionals independent of the center of mass position of the string, rank N projection operators can be identified with solutions of the matter equation describing N space-filling D25-branes. Since all such projection operators are equivalent under unitary transformations, all these solutions are gauge-equivalent in the string field theory. Projection operators described by functionals which are localized in some subset of space-time directions correspond to lower-dimensional D-branes. We showed that the sliver state describing a D-instanton can be modified by a simple change in functional form so that it is only localized in a subset of the space-time dimensions, and thus describes a higher-dimensional Dp-brane. In section 2 we review the formulation of Witten's cubic open string field theory and the results of (I). We explain how the $\star$ product can be realized in the matter sector as matrix (operator) multiplication and give the precise map between this approach and the usual Fock space approach. Some details of this correspondence which did not appear in (I) are relegated to appendices A, B.

In section 3 we turn to the split string formalism in the ghost sector. The situation here is trickier than in the matter sector for several reasons. First, the usual description of ghosts as fermionic operators $c(\sigma), b(\sigma)$, does not lend itself easily to a natural right-left split. The reason is that the $\star$ product does not correspond to a simple overlap. In the matter sector the $\star$ product, $\Psi_1 \star \Psi_2$, equates the right side of Ψ_1 with the left side of Ψ_2 ($x_1(\sigma) = x_2(\pi - \sigma)$). However, for functionals of $c(\sigma)$ the $\star$ product enforces the condition $c_1(\sigma) = -c_2(\pi - \sigma)$. Witten, in his original paper on the subject [4], used a bosonized form of the ghosts. The bosonized ghosts, $\phi(\sigma)$, do allow for the interpretation of the $\star$ product as identifying the left and right halves of the string, however the BRST operator is then a non-local function of the bosonic coordinates and hard to deal with. In [30] the bozonized ghosts were used to verify some of the properties of the string action, however the full verification of the axioms [31] reverted to the use of fermionic ghosts, and this is the formalism that has been primarily used in the intervening years. We shall here use the bosonized form of the ghosts. In the bosonized form the fermions correspond to kinks in the bosonic field, and the operator Q creates a kink at which the bosonic ghost field ϕ has a jump discontinuity of magnitude π. This is a source of many subtleties, but we think that these can be systematically controlled.

The other problem that arises in the ghost sector is the string midpoint. There is an anomaly due to the curvature of the Riemann surface describing the three-string vertex. This leads to an extra term in the $\star$ product that inserts ghost number, which is equivalent to ϕ momentum, at the midpoint. The usual Q acts on the midpoint of the string, and this is a source of many anomalies. In [30, 31] it was shown that the anomalies all cancel in the critical dimension; but this subtlety seems to be an obstacle to a simple realization of the $\star$ product in an operator formalism. If, however, we use, following RSZ, a Q formed solely out of ghosts we can construct Q so that *it does not act at the midpoint of the string.* This leads to an enormous simplication and allows us to implement the operator formalism. Remarkably, with this ansatz for the BRST operator, the problematic anomalies are removed from the picture, even if we are not in the critical dimension!

In section 3 we establish the split field formalism in the ghost sector and carefully show that a large class of Q's, constructed purely out of ghosts, satisfy all the axioms of string field theory, when acting on well-behaved string fields, such as those corresponding to states in the full string Hilbert space.

In section 4 we briefly discuss the issue of how to describe the physical, gauge invariant observables of open string field theory. One of the main advantages of formulating string theory as a theory of open strings is that it should be possible to give a precise list of all possible gauge invariant observables without the problems that one inevitably encounters in a closed string — quantum gravitational — formulation. Thus, in the AdS/CFT duality the observables of the boundary gauge theory are apparent; they are correlation functions of local gauge invariant operators, that correspond via the duality to S-matrix elements of the bulk theory. But in addition, there are non-local gauge invariant observables in the gauge theory, namely the expectation values of Wilson loops, that correspond to some other kind of observables in the bulk theory. Since, in principle, all local operators can be recovered from Wilson loops, the complete set of physical observables in the boundary theory are Wilson loops, which are themselves determined by the gauge invariant eigenvalues of the covariant derivative that effects parallel transport of gauge variant fields. In our operator formulation of open string field theory we have a covariant derivative operator, that transforms covariantly under gauge transformations, and therefore we can define a large class of gauge invariant observables (presumably a complete set of such observables) in terms of the eigenvalues of this operator. This is discussed in section 4.

In section 5 we turn to the solution of the equations of motion in the ghost sector. We first discuss solutions in the ghost sector of the projection equation $\Psi_g = \Psi_g \star \Psi_g$. We then use the ghost projector to construct a class of solutions to the full ghost equation of motion $Q\Psi_g + \Psi_g \star \Psi_g = 0$ using pure ghost operators Q of the form described in section 3. In principle, these solutions allow us to describe an arbitrary system of Dp-branes in the RSZ vacuum string field theory, when we combine the ghost solutions with matter Dp-brane solutions constructed in [22, 14]. We find, however, that the action apparently vanishes for the ghost solutions we construct. Furthermore, due to the simplicity of the split string formalism, it seems that for pure ghost operators Q all solutions of the equations of motion have vanishing action in this formalism. We discuss possible resolutions of this puzzle. We also discuss a candidate class of solutions to Witten's original cubic string field theory with the BRST operator Q_B. These solutions are related to a solution of the form $Q_l|I\rangle$ proposed in [16], which takes the theory to a purely cubic action, where, however, the identity state $|I\rangle$ is replaced by a rank one projector. It seems possible that the solutions based on a rank one projector correspond to single D-branes, while the solution based on the identity string field corresponds to an infinite stack of space-filling D25-branes. While the action for all these solutions also formally vanishes in the split string language, there are additional subtleties in this case arising from the fact that the BRST operator has nontrivial action at the string midpoint, so that the split string formalism is more complicated than in the case of the simple ghost operators we describe in section 3. We speculate that these additional complications may lead to a finite value of the action for these solutions.

Finally, we present some conclusions and open problems in section 6.

2. Review of string field theory

In this section we review some basic aspects of Witten's cubic string field theory [4]. In subsection 2.1 we give a general discussion of the theory using the formal language of functional integrals, and in subsection 2.2 we describe the matter sector of the theory in terms of Fourier modes on the string. In 2.3 we describe the matter sector of the theory in terms of split string degrees of freedom. Appendices A, B describe in detail the connection between normalization factors in the functional integral and Fock space descriptions of the theory.

2.1 Witten's cubic string field theory

In [4], Witten proposed a simple formulation of open bosonic string field theory based on an action of Chern-Simons form

$$S = -\frac{1}{2}\int \Psi \star Q\Psi - \frac{g}{3}\int \Psi \star \Psi \star \Psi\,, \tag{2.1}$$

where g is the open string coupling and Ψ is a string field taking values in a graded algebra $\mathcal{A}$. Associated with the algebra $\mathcal{A}$ there is a star product

$$\star : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}\,, \tag{2.2}$$

under which the degree G is additive ($G_{\Psi\star\Phi} = G_\Psi + G_\Phi$). There is also a BRST operator

$$Q : \mathcal{A} \to \mathcal{A}\,, \tag{2.3}$$

of degree one ($G_{Q\Psi} = 1 + G_\Psi$). String fields can be integrated using

$$\int : \mathcal{A} \to \mathbb{C}\,. \tag{2.4}$$

This integral vanishes for all Ψ with degree $G_\Psi \neq 3$.

The elements $Q, \star, \int$ defining the string field theory are assumed to satisfy the following axioms:

(a) Nilpotency of Q: $Q^2\Psi = 0 \;\forall \Psi \in \mathcal{A}$.

(b) $\int Q\Psi = 0 \;\forall \Psi \in \mathcal{A}$.

(c) Derivation property of Q: $Q(\Psi \star \Phi) = (Q\Psi)\star\Phi + (-1)^{G_\Psi}\Psi\star(Q\Phi) \;\forall \Psi, \Phi \in \mathcal{A}$.

(d) Cyclicity: $\int \Psi \star \Phi = (-1)^{G_\Psi G_\Phi} \int \Phi \star \Psi \;\forall \Psi, \Phi \in \mathcal{A}$.

When these axioms are satisfied, the action (2.1) is invariant under the gauge transformations

$$\delta\Psi = Q\Lambda + \Psi\star\Lambda - \Lambda\star\Psi \tag{2.5}$$

for any gauge parameter $\Lambda \in \mathcal{A}$.

Witten presented this formal structure in [4] and argued that all the needed axioms are satisfied when $\mathcal{A}$ is taken to be the space of string fields

$$\mathcal{A} = \{\Psi[x(\sigma); c(\sigma), b(\sigma)]\}\,, \tag{2.6}$$

which can be described as functionals of the matter, ghost and antighost fields describing an open string in 26 dimensions with $0 \leq \sigma \leq \pi$. For this string field theory, the BRST operator is the usual open string BRST operator of the form

$$Q_B = \int_0^\pi d\sigma\; c(\sigma)\left(T^{(m)}(\sigma) + \frac{1}{2}T^{(g)}(\sigma)\right). \tag{2.7}$$

The star product $\star$ is defined by gluing the right half of one string to the left half of the other using a delta function interaction. The star product factorizes into separate matter and ghost parts. For the matter fields, the star product is given by

$$(\Psi\star\Phi)\,[z(\sigma)] \equiv \int \prod_{0\leq\sigma\leq\frac{\pi}{2}} dy(\sigma)\; dx(\pi-\sigma) \prod_{\frac{\pi}{2}\leq\sigma\leq\pi} \delta[x(\sigma) - y(\pi-\sigma)]\; \Psi[x(\sigma)]\Phi[y(\sigma)] \tag{2.8}$$

$$\begin{aligned} z(\sigma) &= x(\sigma) \qquad \text{for } 0 \leq \sigma \leq \frac{\pi}{2}\,, \\ z(\sigma) &= y(\sigma) \qquad \text{for } \frac{\pi}{2} \leq \sigma \leq \pi\,. \end{aligned} \tag{2.9}$$

Similarly, the integral over a string field factorizes into matter and ghost parts, and in the matter sector is given by

$$\int \Psi = \int \prod_{0 \leq \sigma \leq \pi} dx(\sigma) \prod_{0 \leq \sigma \leq \frac{\pi}{2}} \delta[x(\sigma) - x(\pi - \sigma)] \, \Psi[x(\sigma)] \,. \tag{2.10}$$

The ghost sector of the theory is defined in a similar fashion, but has an anomaly due to the curvature of the Riemann surface describing the three-string vertex. The ghost sector can be described either in terms of fermionic ghost fields $c(\sigma), b(\sigma)$ or through bosonization in terms of a single bosonic scalar field $\phi(\sigma)$. From the functional point of view of (2.9), (2.10), it is easiest to describe the ghost sector in the bosonized language. In this language, the star product in the ghost sector is given by (2.9) with an extra insertion of $\exp(3i\phi(\pi/2)/2)$ inside the integral. Similarly, the integration of a string field in the ghost sector is given by (2.10) with an insertion of $\exp(-3i\phi(\pi/2)/2)$ inside the integral.

The expressions (2.9), (2.10) may seem rather formal, however they can be given precise meaning when described in terms of creation and annihilation operators acting on the string Fock space. This was done explicitly in [30]–[34], where the star product and two- and three-string vertices were described in terms of matter and ghost raising and lowering operators a_n^μ, b_n, c_n. In the following two subsections we describe the matter part of the string field theory using a Fourier mode decomposition which replaces the continuous set of degrees of freedom in the string $x(\sigma)$ with a set of modes x_n. We describe the exact relationship between this approach and the usual Fock space description of the theory in appendices A and B.

2.2 Mode decomposition of matter fields

We can expand each of the 26 matter fields x^μ in Fourier modes through

$$x(\sigma) = x_0 + \sqrt{2} \sum_{n=1}^{\infty} x_n \cos(n\sigma) \,. \tag{2.11}$$

(We will drop most spatial indices in this paper for clarity.) The modes in (2.11) are related to creation and annihilation operators through

$$\begin{aligned} x_n &= \frac{i}{\sqrt{2n}} \left(a_n - a_n^\dagger\right) & p_n &= -i\frac{\partial}{\partial x_n} = \sqrt{\frac{n}{2}} \left(a_n + a_n^\dagger\right) \\ a_n &= -i\left(\sqrt{\frac{n}{2}} x_n + \frac{1}{\sqrt{2n}} \frac{\partial}{\partial x_n}\right) & a_n^\dagger &= i\left(\sqrt{\frac{n}{2}} x_n - \frac{1}{\sqrt{2n}} \frac{\partial}{\partial x_n}\right), \end{aligned} \tag{2.12}$$

for $n \neq 0$, and through

$$\begin{aligned} x_0 &= \frac{i}{2} \left(a_0 - a_0^\dagger\right) & p_0 &= -i\frac{\partial}{\partial x_0} = \left(a_0 + a_0^\dagger\right) \\ a_0 &= -i\left(x_0 + \frac{1}{2} \frac{\partial}{\partial x_0}\right) & a_0^\dagger &= i\left(x_0 - \frac{1}{2} \frac{\partial}{\partial x_0}\right), \end{aligned} \tag{2.13}$$

for the zero modes. We write

$$|x) = \frac{i}{\sqrt{2}} E[|a) - |a^\dagger)]$$

$$|p) = \frac{1}{\sqrt{2}E}[|a) + |a^\dagger)]\,, \tag{2.14}$$

where

$$[a_n, a_m^\dagger] = \delta_{nm}\,, \qquad E^{-1}_{nm} = \delta_{nm}\sqrt{n} + \delta_{n0}\delta_{m0}\sqrt{2}\,. \tag{2.15}$$

We can describe a string field (in the matter sector) as a functional $\Psi[\{x_n\}]$ of the countable set of modes of the matter fields. For such functionals, we can define the string field integral and star product so that

$$\int \Psi = \int \prod_{n=0}^{\infty} dx_n \prod_{k=0}^{\infty} \delta(x_{2k+1}) \Psi[\{x_n\}] \tag{2.16}$$

and

$$\int \Psi \star \Phi = \int \prod_{n=0}^{\infty} dx_n \Psi[\{x_n\}] \Phi[\{(-1)^n x_n\}]\,. \tag{2.17}$$

Throughout the paper we will use (2.16), (2.17) to define the normalization of the string field integral $\int$ and star product $\star$.

2.3 Split string description of matter fields

In this subsection we describe the splitting of the matter fields into left and right components, as developed in [30, 14]. Related approaches were developed in [18, 19, 21, 22].

We split the string coordinate $x(\sigma)$ (which satisfies Neumann boundary conditions at $\sigma = 0, \pi$) into its left and right pieces, according to

$$l(\sigma) = x(\sigma)\,, \quad r(\sigma) = x(\pi - \sigma) \qquad \text{for } 0 \leq \sigma \leq \frac{\pi}{2}\,, \tag{2.18}$$

where $l(\sigma)$ and $r(\sigma)$ obey Neumann boundary conditions at $\sigma = 0$ and Dirichlet boundary conditions at $\sigma = \pi/2$. We can perform a separate mode expansion on the left and right pieces of the string

$$l(\sigma) = \sqrt{2} \sum_{n=0}^{\infty} l_{2n+1} \cos(2n+1)\sigma\,, \tag{2.19}$$

$$r(\sigma) = \sqrt{2} \sum_{n=0}^{\infty} r_{2n+1} \cos(2n+1)\sigma\,. \tag{2.20}$$

The half-string modes are related to the full-string modes through

$$l_{2k+1} = x_{2k+1} + \sum_{n=0}^{\infty} X_{2k+1,2n}\, x_{2n}\,,$$

$$r_{2k+1} = -x_{2k+1} + \sum_{n=0}^{\infty} X_{2k+1,2n}\, x_{2n}\,. \tag{2.21}$$

and

$$
\begin{aligned}
x_{2n+1} &= \frac{1}{2}\left(l_{2n+1} - r_{2n+1}\right), \\
x_{2n} &= \frac{1}{2}\sum_{k=0}^{\infty} X_{2n,2k+1}\left(l_{2k+1} + r_{2k+1}\right),
\end{aligned}
\tag{2.22}
$$

where the transformation matrices $X_{2n,2k+1}, X_{2k+1,2n}$ are given by

$$
\begin{aligned}
X_{2k+1,2n} = X_{2n,2k+1} &= \frac{4(-1)^{k+n}(2k+1)}{\pi\left((2k+1)^2 - 4n^2\right)} \qquad (n \neq 0), \\
X_{2k+1,0} = X_{0,2k+1} &= \frac{2\sqrt{2}(-1)^k}{\pi(2k+1)}.
\end{aligned}
\tag{2.23}
$$

The matrix

$$
X \equiv \begin{pmatrix} 0 & X_{2k+1,2n} \\ X_{2n,2k+1} & 0 \end{pmatrix}
\tag{2.24}
$$

is symmetric and orthogonal: $X = X^T = X^{-1}$.

Using (2.22), (2.21) we can rewrite a string field $\Psi[\{x_n\}]$ as a functional of the right- and left-half string degrees of freedom $\Psi[\{l_{2k+1}\};\{r_{2k+1}\}]$. Since $\det X = 1$, we can then write the string field integral (2.16) as

$$
\begin{aligned}
\int \Psi &= \int \prod_{k=0}^{\infty}\left(\frac{1}{2}\, dl_{2k+1}\, dr_{2k+1}\, \delta\left(\frac{l_{2k+1} - r_{2k+1}}{2}\right)\right)\Psi[\{l_{2k+1}\};\{r_{2k+1}\}] \\
&= \int \prod_{k=0}^{\infty} dl_{2k+1}\; \Psi[\{l_{2k+1}\};\{l_{2k+1}\}].
\end{aligned}
\tag{2.25}
$$

Similarly, the star product $\Psi \star \Phi$ is given in the split string language by

$$
(\Psi\star\Phi)[\{l_{2k+1}\};\{r_{2k+1}\}] = \int \prod_{k=0}^{\infty} dy_{2j+1}\, \Psi[\{l_{2k+1}\};\{y_{2j+1}\}]\Phi[\{y_{2j+1}\};\{r_{2k+1}\}].
\tag{2.26}
$$

This gives us a complete formulation of the matter part of the cubic string field theory in terms of modes on the half-string. In this formulation, the string field $\Psi[l;r]$ essentially acts as an operator on a space of half-string functionals

$$
\Psi \Rightarrow \hat{\Psi}.
\tag{2.27}
$$

In this operator language, the string field integral (2.25) and star product (2.26) can be described in terms of a trace and operator multiplication respectively

$$
\begin{aligned}
\int \Psi &\Rightarrow \operatorname{Tr}\hat{\Psi} \\
\Psi\star\Phi &\Rightarrow \hat{\Psi}\hat{\Phi}.
\end{aligned}
\tag{2.28}
$$

One subtle aspect of this split string formalism involves the role of the string midpoint. As we will discuss further in section 3, these issues are more significant in the ghost sector of the theory. One way of dealing with the midpoint is to explicitly subtract out the midpoint before using (2.21) to relate the full-string modes to half-string modes. We have then

$$\begin{aligned} \tilde{x}(\sigma) &= x(\sigma) - \bar{x} \\ \tilde{x}_n &= x_n - \delta_{n0}\bar{x} \end{aligned} \tag{2.29}$$

where

$$\bar{x} = x(\pi/2) = x_0 + \sqrt{2}\sum_{n=1}^{\infty}(-1)^n x_{2n} \tag{2.30}$$

is the string midpoint. Plugging $\tilde{x}_n$ into (2.21) gives us a set of left and right modes $\tilde{l}_{2k+1}, \tilde{r}_{2k+1}$ associated with a string having vanishing midpoint. This gives us a representation of the string field Ψ as an operator-valued function on space-time

$$\Psi \Rightarrow \hat{\Psi} = \tilde{\Psi}(\bar{x}) \tag{2.31}$$

where at each point $\bar{x}$, the operator $\tilde{\Psi}(\bar{x})$ acts on the usual Hilbert space of states of a string with Neumann-Dirichlet boundary conditions. For a string field of definite momentum p_0, the associated operator-valued function takes the form

$$\tilde{\Psi}(\bar{x}) = e^{ip_0\bar{x}}\tilde{\Psi}_0 \tag{2.32}$$

where $\tilde{\Psi}_0$ is a single operator acting on the ND Hilbert space. We will find this description of string fields with definite momentum useful in the later discussion of the ghost sector of the theory.

A class of matter string fields of particular interest are those which satisfy the projection equation

$$\Psi = \Psi \star \Psi\,. \tag{2.33}$$

As discussed in [28], such states could be a first step towards an analytic construction of a solution to the full string field equation of motion $Q\Psi + g\Psi\star\Psi = 0$. Furthermore, in the vacuum string field theory postulated by Rastelli, Sen, and Zwiebach, such states correspond to D-brane solitons [24, 27]. In [14, 22] it was shown that the projection operators associated with single D-branes are rank one projectors on the appropriate space of string functionals when described in the split string language; multiple D-brane configurations are described by higher-rank projections. In the notation we are using here, a simple class of rank one projectors take the form

$$\Psi[l;r] = \left(\det\frac{M}{\pi}\right)^{26/2}\exp\left(\sum_{k=0}^{\infty} -\frac{1}{2}l_{2k+1}M_{2k+1,2j+1}l_{2j+1} - \frac{1}{2}r_{2k+1}M_{2k+1,2j+1}r_{2j+1}\right). \tag{2.34}$$

The matrix M giving one such state associated with a D-instanton was explicitly analyzed in [14] and related to the "sliver state" originally found in [28, 29]. From (2.26) we see that the state (2.34) is indeed a projection operator satisfying (2.33), which is $\hat{\Psi}\hat{\Psi} = \hat{\Psi}$ in the operator language. From (2.25) we see that furthermore $\mathrm{Tr}\,\hat{\Psi} = 1$ for the state (2.34), so that this is actually a rank one projection operator. In appendices A and B we derive the precise relations between the normalization factors in the functional and Fock space approaches, and clarify the relationship between the normalization we use here for projection operators and that used in the discussion of [22].

3. Split string ghosts

We now discuss the split string approach to the ghost sector of the theory. As discussed in the introduction, there are several complications which make the ghost sector of the theory more difficult to treat in this language than the matter sector. In the fermionic language, the overlap condition and anomalous midpoint insertions are rather difficult to treat in a simple fashion. These problems become much simpler in the bosonized formulation of the ghost sector, in which the ghosts are associated with a single additional bosonic field with half-integral momentum. In this section we give a detailed discussion of the bosonized description of the ghost sector, and show that the split string formalism of the matter sector carries over immediately to the split string description of the string field star product $\star$ and integral $\int$ for the bosonized ghosts. The remaining complication is the BRST operator Q. In this paper we concentrate on a particularly simple class of pure ghost BRST operators which have no action on the string midpoint, and for which the split string description is therefore particularly simple. This is the class of ghost operators which was proposed for the vacuum string field theory in [24]. We show that these operators satisfy all the axioms of string field theory when acting on suitably well-behaved string states.

In subsection 3.1 we describe in detail the bosonization of the fermionic ghost sector. We give explicit formulae relating the ghost fields in the two formalisms, and we discuss a regulator that controls certain divergences which appear when doing calculations with ghost fields. In subsection 3.2 we discuss the star product and string field integral in the bosonized ghost language. These operations are exactly the same as those we have discussed in the matter sector except for the insertion of a midpoint-dependent phase factor. In 3.3 we describe in detail the class of pure ghost BRST operators of interest, and show that these operators indeed obey the axioms of string field theory when acting on states in the original string Fock space.

3.1 Ghost bosonization

In this section we review some basic aspects of the bosonization of the ghosts in the open bosonic string. Most of this material is described in [35]

3.1.1 Fermionic ghosts

The ghost and antighost fields on the open string satisfy periodic boundary conditions

$$c^{\pm}(\sigma + 2\pi) = c^{\pm}(\sigma)\,, \qquad b_{\pm}(\sigma + 2\pi) = b_{\pm}(\sigma)\,. \tag{3.1}$$

These Grassmann fields have mode decompositions

$$\begin{aligned} c^{\pm}(\sigma) &= \sum_n c_n e^{\mp in\sigma} \\ b_{\pm}(\sigma) &= \sum_n b_n e^{\mp in\sigma}\,. \end{aligned} \tag{3.2}$$

The ghost creation and annihilation operators satisfy

$$\{c_n, b_m\} = \delta_{n+m,0}\,, \quad \{c_n, c_m\} = \{b_n, b_m\} = 0\,. \tag{3.3}$$

The ghost Fock space has a pair of vacua $|\pm\rangle$ annihilated by c_n, b_n for $n > 0$. These two vacua satisfy

$$\begin{aligned} c_0|-\rangle &= |+\rangle \qquad & c_0|+\rangle &= 0 \\ b_0|+\rangle &= |-\rangle \qquad & b_0|-\rangle &= 0\,. \end{aligned}$$

We shall define the Fock space so that b_0, c_0 are hermitian and $c_n^\dagger = c_{-n}$, $b_n^\dagger = b_{-n}$. It follows that $\langle +|+\rangle = \langle +|c_0|-\rangle = 0$, and similarly $\langle -|-\rangle = 0$. We normalize the vacua so that

$$\langle +|b_0|+\rangle = \langle -|c_0|-\rangle = 1\,. \tag{3.4}$$

The ghost number is

$$G = \sum_{n=1}^{\infty} [c_{-n}b_n - b_{-n}c_n] + \frac{1}{2}[c_0 b_0 - b_0 c_0] + \frac{3}{2} \tag{3.5}$$

so that the c_n (b_n) have ghost number 1 (-1), for all n. The vacua $|+\rangle$ and $|-\rangle$ have ghost number 2 and 1 respectively.

3.1.2 Bosonized ghosts

An alternative description of the fermionic ghost fields can be given in terms of a single periodic bosonic field

$$\phi(\sigma) = \phi_0 + \sqrt{2}\sum_{n=1}^{\infty} \phi_n \cos(n\sigma)\,. \tag{3.6}$$

This field can be decomposed into right-moving and left-moving parts $\phi^{\pm}$ and modes $a_n, a_n^\dagger$ through

$$\begin{aligned} \phi^{\pm}(\sigma) &= \phi_0 \pm \sigma p_0 + \sum_{n=1}^{\infty}\left[\sqrt{2}\phi_n \cos(n\sigma) \pm \frac{\sqrt{2}p_n}{n}\sin(n\sigma)\right] \\ &= \phi_0 \pm \sigma p_0 + i\sum_{n=1}^{\infty}\left[-\frac{1}{\sqrt{n}}e^{\pm in\sigma}a_n^\dagger + \frac{1}{\sqrt{m}}e^{\mp im\sigma}a_m\right] \end{aligned} \tag{3.7}$$

where we use the conventions of section 2 for the definitions of $a_n, a_n^\dagger$, and p_n.

$$[a_n, a_m^\dagger] = \delta_{n,m}, \quad \phi_n = \frac{i}{\sqrt{2n}}(a_n - a_n^\dagger), \quad p_n = \sqrt{\frac{n}{2}}(a_n + a_n^\dagger) = -i\frac{\partial}{\partial \phi_n}. \tag{3.8}$$

We will often write $a_{-n} = a_n^\dagger$.

The fermionic ghosts are related to the bosonized ghost field through the bosonization formulae [36, 35, 4]

$$c^+(\sigma) = :e^{i\phi^+(\sigma)}:= e^{i\phi_0} e^{i\sigma(p_0+1/2)} e^{\sum_{n=1}^\infty \frac{1}{\sqrt{n}} e^{in\sigma} a_n^\dagger} e^{-\sum_{m=1}^\infty \frac{1}{\sqrt{m}} e^{-im\sigma} a_m} \tag{3.9}$$

$$c^-(\sigma) = :e^{i\phi^-(\sigma)}:= e^{i\phi_0} e^{-i\sigma(p_0+1/2)} e^{\sum_{n=1}^\infty \frac{1}{\sqrt{n}} e^{-in\sigma} a_n^\dagger} e^{-\sum_{m=1}^\infty \frac{1}{\sqrt{m}} e^{im\sigma} a_m} \tag{3.10}$$

$$b_+(\sigma) = :e^{-i\phi^+(\sigma)}:= e^{-i\phi_0} e^{-i\sigma(p_0-1/2)} e^{-\sum_{n=1}^\infty \frac{1}{\sqrt{n}} e^{in\sigma} a_n^\dagger} e^{\sum_{m=1}^\infty \frac{1}{\sqrt{m}} e^{-im\sigma} a_m} \tag{3.11}$$

$$b_-(\sigma) = :e^{-i\phi^-(\sigma)}:= e^{-i\phi_0} e^{i\sigma(p_0-1/2)} e^{-\sum_{n=1}^\infty \frac{1}{\sqrt{n}} e^{-in\sigma} a_n^\dagger} e^{\sum_{m=1}^\infty \frac{1}{\sqrt{m}} e^{im\sigma} a_m} \tag{3.12}$$

The appearance of the extra factors of $e^{\pm i\sigma/2}$ in these formulae indicates that for $c^\pm, b_\pm$ to be periodic we must consider states described by functionals $\Psi[\phi(\sigma)]$ with half-integral p_0, so that

$$\Psi[\phi(\sigma) + 2\pi] = -\Psi[\phi(\sigma)]. \tag{3.13}$$

From these formulae we can compute the anticommutators

$$\{c^+(\sigma), c^+(\tau)\} = 0 \tag{3.14}$$

$$\{c^-(\sigma), c^-(\tau)\} = \{b_+(\sigma), b_+(\tau)\} = \{b_-(\sigma), b_-(\tau)\} = 0 \tag{3.15}$$

$$\{c^+(\sigma), b_+(\tau)\} = 2\pi\delta(\sigma - \tau). \tag{3.16}$$

There are several approaches one can take to computing the anticommutators (3.14)–(3.16). We now describe one of these approaches in detail for the relation (3.16), illustrating a method of regulating divergences which we will use throughout the paper.

In many computations involving operators like (3.9) divergent infinite sums arise. One way of regulating these sums is to replace the canonical commutation relations for the modes $[a_n, a_m^\dagger] = \delta_{n,m}$ with the commutation relations

$$\left[a_n, a_m^\dagger\right] = x^n \delta_{n,m} \tag{3.17}$$

where $x = 1 - \epsilon < 1$. With this regulator $[p_n, \phi_n] = -ix^n$, so the regulated form of p_n is $p_n \to -ix^n\partial/\partial\phi_n$, which amounts to replacing $p_n \to x^n p_n$. Using this regulator, from (3.9), (3.11) we have

$$\{c^+(\sigma), b_+(\tau)\} = \left(\frac{e^{\frac{i}{2}(\sigma-\tau)}}{1 - xe^{i(\sigma-\tau)}} + \frac{e^{\frac{i}{2}(\tau-\sigma)}}{1 - xe^{i(\tau-\sigma)}}\right) : c^+(\sigma) b_+(\tau): \tag{3.18}$$

where

$$: c^+(\sigma) b_+(\tau) := e^{ip_0(\sigma-\tau)} e^{\sum_{n=1}^\infty \frac{1}{\sqrt{n}}(e^{in\sigma} - e^{in\tau}) a_n^\dagger} e^{\sum_{m=1}^\infty \frac{1}{\sqrt{m}}(e^{-im\tau} - e^{-im\sigma}) a_m}. \tag{3.19}$$

The quantity in parentheses in (3.18) clearly vanishes as $x \to 1$ when $\sigma \neq \tau$. Integrating this quantity, we see that as $x \to 1$, we have

$$\int_{\sigma=\tau-\pi}^{\sigma=\tau+\pi} \left(\frac{e^{\frac{i}{2}(\sigma-\tau)}}{1 - xe^{i(\sigma-\tau)}} + \frac{e^{\frac{i}{2}(\tau-\sigma)}}{1 - xe^{i(\tau-\sigma)}} \right) = \frac{-4i}{\sqrt{x}} \ln \left(\frac{1+i\sqrt{x}}{1-i\sqrt{x}} \right) \to 2\pi \,. \tag{3.20}$$

Since (3.19) goes to 1 at $\sigma = \tau$, we see that the anticommutator (3.18) goes to $2\pi\delta(\sigma - \tau)$ in the limit $x \to 1$. Note that this argument involves some implicit assumptions about the nature of the state the operator is acting on. When (3.19) acts on a string functional which is outside the Fock space and is sufficiently poorly behaved, it is possible that the limit of this operator as $x \to 1$ may not give 1. Technically, relations (3.14)–(3.16) are only valid when these operators act on a well-behaved string functional such as those in the Hilbert space associated with the bosonized ghost field. We will return to this point in subsection 3.3.

From (3.9)–(3.12) we can expand in modes to find an expression for how any fermionic mode c_n, b_m acts on a state of fixed ghost number in terms of the bosonic modes a_n. This expression depends upon the ghost number of the state being acted on. For example, acting on states with $p_0 = -1/2$ we have

$$c_0 = e^{i\phi_0} \left[1 + a_{-1}a_1 + \frac{1}{2\sqrt{2}} a_{-1}^2 a_2 + \frac{1}{2\sqrt{2}} a_{-2}a_1^2 + \frac{1}{2} a_{-2}a_2 + \cdots \right] . \tag{3.21}$$

Acting on states with $p_0 = 1/2$, on the other hand, $c_0 = e^{i\phi_0}[a_1 + \cdots]$. From these expressions, the action of the ghost and antighost modes on any Fock space state can be computed. Furthermore, the correspondence between states in the bosonized and fermionic ghost Fock spaces can be determined in this fashion. For example, we have

$$\begin{aligned} |-\rangle \quad &\leftrightarrow \quad |p_0 = -1/2\rangle \\ |+\rangle = c_0|-\rangle \quad &\leftrightarrow \quad |p_0 = 1/2\rangle \\ b_{-1}|-\rangle \quad &\leftrightarrow \quad |p_0 = -3/2\rangle \,. \end{aligned} \tag{3.22}$$

The bosonic and fermionic representations of the ghosts can also be related using the formula

$$J_+(\sigma) = \partial_\sigma \phi^+(\sigma) = \lim_{\tau \to \sigma} \left[c^+(\sigma) b_+(\tau) + \frac{i}{\sigma - \tau} \right] , \tag{3.23}$$

from which it follows that

$$\begin{aligned} a_n &= \frac{1}{\sqrt{|n|}} \sum_m c_m b_{n-m} \qquad , n \neq 0 \\ p_0 &= \sum_{n=1}^{\infty} [c_{-n}b_n - b_{-n}c_n] + \frac{1}{2} [c_0 b_0 - b_0 c_0] \,. \end{aligned} \tag{3.24}$$

Note that $G = p_0 + 3/2$.

We will often find it convenient to express the ghost fields in terms of the modes ϕ_n and derivatives $\partial/\partial\phi_n$. Using the relations

$$
\begin{aligned}
&\exp\left[\frac{1}{\sqrt{n}}e^{in\sigma}a_n^\dagger\right]\exp\left[-\frac{1}{\sqrt{n}}e^{-in\sigma}a_n\right] = \\
&\quad = \exp\left[\frac{1}{2n} - i\frac{\sin(2n\sigma)}{2n}\right]\exp\left[\sqrt{2}\,\frac{\sin(n\sigma)}{n}\frac{\partial}{\partial\phi_n}\right]\exp\left[\sqrt{2}i\cos(n\sigma)\phi_n\right] \\
&\quad = \exp\left[\frac{1}{2n} + i\frac{\sin(2n\sigma)}{2n}\right]\exp\left[\sqrt{2}i\cos(n\sigma)\phi_n\right]\exp\left[\sqrt{2}\,\frac{\sin(n\sigma)}{n}\frac{\partial}{\partial\phi_n}\right],
\end{aligned}
\tag{3.25}
$$

$c^\pm(\sigma)$, for example, can be written as

$$
c^\pm(\sigma) = K e^{i\epsilon(\sigma)\frac{\pi}{4}} e^{\pm i\sigma(p_0-1)} e^{i\phi(\sigma)} e^{\pm\sum_{n=1}^\infty \sqrt{2}\,\frac{\sin(n\sigma)}{n}\frac{\partial}{\partial\phi_n}} \tag{3.26}
$$

$$
= K e^{-i\epsilon(\sigma)\frac{\pi}{4}} e^{\pm i\sigma p_0} e^{\pm\sum_{n=1}^\infty \sqrt{2}\,\frac{\sin(n\sigma)}{n}\frac{\partial}{\partial\phi_n}} e^{i\phi(\sigma)}, \tag{3.27}
$$

where we have used

$$
\sum_{n=1}^{\infty}\frac{\sin(2n\sigma)}{2n} = \frac{\pi}{4}\epsilon(\sigma) - \frac{\sigma}{2} \qquad \text{for } -\pi \le \sigma \le \pi\,, \tag{3.28}
$$

and K is an (infinite) constant, $K = \exp[\sum_n 1/2n]$. In (3.28) we have kept the factor of $\epsilon(\sigma) = \pm 1$, for $\sigma > 0, \sigma < 0$, since we will use this formula sometimes for negative values of σ. Using the regulator (3.17), this constant becomes

$$
K_x = \exp\left(\sum_{n=1}^{\infty}\frac{x^n}{2n}\right) = \frac{1}{\sqrt{1-x}}\,. \tag{3.29}
$$

The regulated form of (3.28) is

$$
\sum_{n=1}^{\infty}\frac{\sin(2n\sigma)x^n}{2n} = -\frac{\sigma}{2} + \frac{1}{4i}\ln\left[\frac{e^{i\sigma} - e^{-i\sigma} + \epsilon\, e^{-i\sigma}}{e^{-i\sigma} - e^{i\sigma} + \epsilon e^{i\sigma}}\right]. \tag{3.30}
$$

This expression gives (3.28) when $\epsilon \ll \sigma$. For small $\epsilon \sim \sigma$, we can replace this with

$$
\sum_{n=1}^{\infty}\frac{\sin(2n\sigma)x^n}{2n} \sim -\frac{\sigma}{2} + \frac{1}{4i}\ln\left[\frac{\epsilon + 2i\sigma}{\epsilon - 2i\sigma}\right] \tag{3.31}
$$

for a simpler but equivalent regulator. The regulated form of (3.26), (3.27) is then

$$
c^\pm(\sigma) = K_x\left(\frac{\epsilon + 2i\sigma}{\epsilon - 2i\sigma}\right)^{1/4} e^{\pm i\sigma(p_0-1)} e^{i\phi(\sigma)} e^{\pm\sum_{n=1}^\infty \sqrt{2}\,\frac{\sin(n\sigma)}{n}x^n\frac{\partial}{\partial\phi_n}} \tag{3.32}
$$

$$
= K_x\left(\frac{\epsilon - 2i\sigma}{\epsilon + 2i\sigma}\right)^{1/4} e^{\pm i\sigma p_0} e^{\pm\sum_{n=1}^\infty \sqrt{2}\,\frac{\sin(n\sigma)}{n}x^n\frac{\partial}{\partial\phi_n}} e^{i\phi(\sigma)}, \tag{3.33}
$$

Using (3.27) we can evaluate the action of $c^+(\sigma_0)$ on a position basis state $|\phi(\sigma)\rangle = |\{\phi_n\}\rangle$, which is given by (here we set $x = 1$)

$$\begin{aligned} c^+(\sigma_0)|\phi(\sigma)\rangle &= Ke^{-i\epsilon(\sigma_0)\frac{\pi}{4}}e^{i\phi(\sigma_0)}e^{i\sigma_0 p_0}|\phi(\sigma) - (\pi - \sigma_0)\theta(\sigma_0 - \sigma) + \sigma_0\theta(\sigma - \sigma_0)\rangle \\ &= Ke^{-i\epsilon(\sigma_0)\frac{\pi}{4}}e^{i\phi(\sigma_0)}|\phi(\sigma) - \pi\theta(\sigma_0 - \sigma)\rangle\,, \end{aligned} \tag{3.34}$$

where we have used:

$$2\sum_{n=1}^{\infty}\frac{\sin(n\sigma_0)\cos(n\sigma)}{n} = (\pi - \sigma_0)\theta(\sigma_0 - \sigma) - \sigma_0\theta(\sigma - \sigma_0)\,, \qquad 0 \le \sigma_0, \sigma \le \pi\,. \tag{3.35}$$

Note that in this equation, we must interpret $\theta(0)$ as equal to 1/2, since when $\sigma_0 = \sigma$, (3.35) is identical to twice (3.28) Similarly to (3.34), we have

$$c^-(\sigma_0)|\phi(\sigma)\rangle = Ke^{-i\epsilon(\sigma_0)\frac{\pi}{4}}e^{i\phi(\sigma_0)}|\phi(\sigma) + \pi\theta(\sigma_0 - \sigma)\rangle\,. \tag{3.36}$$

Thus, we see that the operators $c^\pm(\sigma)$ introduce a jump discontinuity of magnitude π in a smooth string configuration. The presence of such kinks is the key mechanism underlying the encoding of the full structure of the fermionic ghost-antighost Fock space in the bosonized ghost Fock space.

We can describe the action of $c^\pm(\sigma_0)$ on a general functional $\Psi[\phi(\sigma)]$ through

$$c^\pm(\sigma_0)\Psi[\phi(\sigma)] = Ke^{i\epsilon(\sigma_0)\frac{\pi}{4}}e^{i\phi(\sigma_0)}\Psi[\phi(\sigma) \pm \pi\theta(\sigma_0 - \sigma)]\,, \tag{3.37}$$

or,

$$c^\pm(\sigma_0)\Psi[\phi_0, \phi_n] = Ke^{i\epsilon(\sigma_0)\frac{\pi}{4}}e^{i\phi(\sigma_0)}\Psi[\phi_0 \pm \sigma_0, \phi_n \pm \sqrt{2}\frac{\sin(n\sigma_0)}{n}]\,. \tag{3.38}$$

If we wish to use the regulated form of $c^\pm$, we need the sum ($\epsilon = 1 - x$)

$$\begin{aligned} 2\sum_{n=1}^{\infty}&\frac{\sin(n\sigma_0)\cos(n\sigma)x^n}{n} = \\ &= -\sigma_0 + \frac{1}{2i}\ln\left[\frac{(e^{i(\sigma_0-\sigma)/2} - xe^{i(\sigma-\sigma_0)/2})(e^{i(\sigma_0+\sigma)/2} - xe^{-i(\sigma+\sigma_0)/2})}{(e^{i(\sigma-\sigma_0)/2} - xe^{i(\sigma_0-\sigma)/2})(e^{-i(\sigma_0+\sigma)/2} - xe^{i(\sigma+\sigma_0)/2})}\right] \\ &\sim \frac{\pi}{2} - \sigma_0 + \frac{1}{2i}\ln\left[\frac{\epsilon + i(\sigma_0 - \sigma)}{\epsilon - i(\sigma_0 - \sigma)}\right], \end{aligned} \tag{3.39}$$

where in the second line we have simplified using $\epsilon \sim \sigma_0 - \sigma \ll \sigma_0$. This expression reduces to (3.35) when $\epsilon \to 0$. The regulated form of (3.37) is

$$c^\pm(\sigma_0)\Psi[\phi(\sigma)] = K_x\left(\frac{\epsilon + 2i\sigma}{\epsilon - 2i\sigma}\right)^{1/4} e^{i\phi(\sigma_0)}\,\Psi\left[\phi(\sigma) \pm \pi/2 \pm \frac{1}{2i}\ln\left(\frac{\epsilon + i(\sigma_0 - \sigma)}{\epsilon - i(\sigma_0 - \sigma)}\right)\right]. \tag{3.40}$$

It might appear disturbing that the action of $c^\pm(\sigma)$ on a functional is proportional to the infinite factor K. However this factor is precisely what is needed to render the action of $c^{\pm}(\sigma)$ finite, when acting on a functional that lies within the ghost Fock space. Namely, when $c^{\pm}(\sigma)$ is smeared with a smooth function, the action of the resulting operator on a Fock space functional gives another Fock space functional. This may seem surprising, since $c^{\pm}(\sigma)$ creates a (fermionic) kink at σ; but when smeared the kinks average out and the factor of K cancels a corresponding vanishing factor of $1/K$.

As an example of this phenomenon it is instructive to consider the relation $c_0|-\rangle = |+\rangle$. From (3.22), we see that in the bosonized language this becomes

$$\frac{1}{2\pi}\int_0^{\pi} d\sigma \left(c^+(\sigma) + c^-(\sigma)\right)|p_0 = -1/2\rangle = |p_0 = 1/2\rangle \,. \tag{3.41}$$

This relation is straightforward to derive using the expressions for $c^{\pm}$ in terms of raising and lowering operators (3.9), (3.10). It is somewhat less transparent, however, when we use (3.37) to describe the action of the c's on the vacuum wavefunction

$$\Psi_0[\phi(\sigma)] = C\exp(-i\phi_0/2)\exp\left(-\sum_{n=1}^{\infty}\frac{n}{2}\phi_n^2\right) \tag{3.42}$$

where

$$C = \prod_n \left(\frac{n}{\pi}\right)^{26/4} . \tag{3.43}$$

We have

$$\begin{aligned}
&c_0\Psi_0[\phi(\sigma)] = \\
&= \frac{1}{2\pi}\int_0^{\pi} d\sigma K C e^{i\epsilon(\sigma)\pi/4}\exp\left(i\frac{\phi_0}{2} + i\sum_n \sqrt{2}\phi_n\cos(n\sigma)\right)\times \\
&\quad\times\left[\exp\left(-i\frac{\sigma}{2} - \sum_n\frac{n}{2}\left(\phi_n + \frac{\sqrt{2}}{n}\sin n\sigma\right)^2\right) + \exp\left(i\frac{\sigma}{2} - \sum_n\frac{n}{2}\left(\phi_n - \frac{\sqrt{2}}{n}\sin n\sigma\right)^2\right)\right] \\
&= C\exp\left(i\frac{\phi_0}{2} - \sum_n\frac{n}{2}\phi_n^2\right)\int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}\exp\left(-i\frac{\sigma}{2} + i\frac{\epsilon(\sigma)\pi}{4} + \sum_n\left(i\sqrt{2}\phi_n e^{in\sigma} + \frac{\cos 2n\sigma}{2n}\right)\right) \\
&= C\exp\left(i\frac{\phi_0}{2} - \sum_n\frac{n}{2}\phi_n^2\right)\int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}\frac{\exp\left(-i\sigma/2 + i\epsilon(\sigma)\pi/4 + \sum_n i\sqrt{2}\phi_n e^{in\sigma}\right)}{(2 - 2\cos 2\sigma)^{1/4}} \\
&= C\exp\left(i\frac{\phi_0}{2} - \sum_n\frac{n}{2}\phi_n^2\right) .
\end{aligned} \tag{3.44}$$

Here we used the result, derived in appendix C, that the integral in the next-to-last line equals one.

Thus we see that c_0, although it is an integral of operators that create kinks, acts on the vacuum functional, which is a Gaussian about $\phi_n = 0$, to produce another Gaussian localized at $\phi_n = 0$. The kinks average out. The same will hold for any state in the Fock space. Such states are of the form $P(\phi_n)\Psi_0[\phi(\sigma)]$, where $P(\phi_n)$ is a polynomial in the ϕ_n's.

3.2 Splitting bosonic ghosts

Using the bosonized formulation of the ghosts, we can use the same approach to splitting the bosonic ghost field as was described in section 2.3 for the matter fields; related work appeared in [20]. We are particularly interested in ghost functionals $\Psi[\phi(\sigma)]$ with fixed ghost number $G = p_0+3/2$. By separating out the string midpoint $\bar{\phi}$ explicitly, as in (2.31), (2.32), such functionals can be written as operator-valued functions on space-time

$$\Psi[\phi(\sigma)] \Rightarrow \hat{\Psi} = e^{ip_0\bar{\phi}}\tilde{\Psi}\,, \tag{3.45}$$

where $\tilde{\Psi}$ acts on the space of functionals $\chi[\tilde{l}]$ of half-string bosonized ghost field configurations $l(\sigma) = \phi(\sigma) - \bar{\phi}, 0 \leq \sigma \leq \pi/2$ vanishing at the midpoint. In some situations we will separate out the midpoint dependence of a state explicitly; in other situations it is more convenient to keep the functional dependence on the midpoint encoded in the left and right degrees of freedom through $\Psi[l;r] \Rightarrow \hat{\Psi}$.

The star product in the ghost sector has an extra insertion of $\exp(3i\bar{\phi}/2)$ due to the ghost current anomaly. Thus, a pair of states A, B associated with operators $\hat{A}, \hat{B}$ through (3.45) have a star product given by

$$A \star B \Rightarrow \hat{A}e^{\frac{3i}{2}\bar{\phi}}\hat{B} = e^{i\bar{\phi}(3/2+p_{0(A)}+p_{0(B)})}\tilde{A}\tilde{B}\,, \tag{3.46}$$

where $p_{0(A,B)} + 3/2$ are the ghost numbers of states A, B respectively. We will find it useful to write the midpoint factor $e^{3i\bar{\phi}/2}$ appearing in this expression as

$$M = \exp\left(\frac{3i\bar{\phi}}{2}\right). \tag{3.47}$$

Since all ghost states of interest have definite ghost number G, and therefore definite ghost momentum p_0 all the midpoint dependence of the corresponding functionals are encoded in (3.45). It follows that M simply adds momentum to a state and therefore commutes with all operators $\hat{A}$ associated with states of definite ghost number. In terms of M, we write

$$A \star B \Rightarrow \hat{A}M\hat{B}\,. \tag{3.48}$$

The integral over a ghost string field $\Psi[\phi(\sigma)]$ is given in this notation by

$$\int \Psi[\phi(\sigma)] \Rightarrow \mathrm{Tr}\, M^{-1}\hat{\Psi} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\bar{\phi}\, \mathrm{Tr}_{\mathrm{ND}}\, \left(M^{-1}\hat{\Psi}\right) \tag{3.49}$$

where the trace $\mathrm{Tr}_{\mathrm{ND}}$ is taken in the ND half-string space spanned by the basis $|\tilde{l}\rangle$.

3.3 Ghostly Q's

We are now interested in constructing a class of operators Q which satisfy the axioms listed in section 2.1 for the cubic string field theory. Following Rastelli, Sen and

Zwiebach [24], we will consider operators which are linear in $c^{\pm}$. We can construct a general such operator of the form

$$Q_f = \int_0^{\pi} d\sigma \left(f_+(\sigma) c^+(\sigma) + f_-(\sigma) c^-(\sigma) \right). \tag{3.50}$$

In order for Q_f to define a consistent cubic string field theory, we must show that

(a) $Q_f^2 = 0$

(b) $\int Q_f \Psi = 0$, $\forall \Psi$

(c) $Q_f(A \star B) = (Q_f A) \star B + (-1)^{G_A} A \star (Q_f B)$, where G_A is the ghost number of A.

We will restrict $f_{\pm}(\sigma)$ to satisfy the conditions

$$f_{\pm}(\sigma) = f_{\pm}(\pi - \sigma) \tag{3.51}$$

$$f_{\pm}(\pi/2) = 0\,. \tag{3.52}$$

The first restriction, $f_{\pm}(\sigma) = f_{\pm}(\pi - \sigma)$, will be necessary if Q_f is to be a derivation and satisfy $\int Q_f \Psi = 0$. We require the second condition $f_{\pm}(\pi/2) = 0$ so as to avoid problems at the midpoint. Thus, for example, we do not consider $f = 1$, which would produce $Q_1 = c_0$, but we do consider $f = 1 + \cos(2\sigma)$, which produces $Q_{1+\cos(2\sigma)} = c_0 + (c_2 + c_{-2})/2$. In [24], Rastelli, Sen and Zwiebach considered two classes of pure ghost BRST operators. The first class, including operators such as c_0, do not annihilate the ghost identity field $|I_g\rangle$, and therefore do not satisfy the condition $\int Q\Psi = 0$. The second class of operators, including operators such as $c_0 + (c_2 + c_{-2})/2$ do annihilate the identity and therefore satisfy the string field theory axiom $\int Q\Psi = 0$. As we will see, the restriction (3.52) amounts to the selection of BRST operators in the second class.

We will now proceed to demonstrate each of the conditions (a-c) for BRST operators given by (3.50) under the restrictions (3.51), (3.52), using the split string formalism described in the last two subsections. We shall not use the regulator in this discussion for clarity, but for a situation in which complications might be imagined we show explicitly in appendix D that the result still follows when the regulator is included, at least when dealing with well-behaved functionals such as those in the string Hilbert space. We will work with a simplified BRST operator with $f_-(\sigma) = 0$; the proofs for a general operator with $f_- \neq 0$ follow in a similar fashion.

We begin by describing the action of Q_f on a state $\Psi[\phi(\sigma)]$ with momentum p_0 explicitly in both the full- and half-string formalisms. From (3.37), we have

$$Q_f \Psi[\phi(\sigma)] = \int_0^{\pi} d\tau\, f_+(\tau) K e^{i\frac{\pi}{4}} e^{i\phi(\tau)} \Psi[\phi(\sigma) + \pi\theta(\tau - \sigma)]\,. \tag{3.53}$$

In terms of the half-string variables, this becomes

$$Q_f \Psi[l(\sigma), r(\sigma)] = \int_0^{\pi/2} d\tau\, f_+(\tau) K e^{i\frac{\pi}{4}} \Big[e^{il(\tau)} \Psi[l(\sigma) + \pi\theta(\tau - \sigma), r(\sigma)] + \qquad (3.54)$$

$$+ e^{ir(\tau)} \Psi[l(\sigma) + \pi, r(\sigma) + \pi\theta(\sigma - \tau)] \Big]$$

$$= \int_0^{\pi/2} d\tau\, f_+(\tau) K e^{i\frac{\pi}{4}} \Big[e^{il(\tau)} \Psi[l(\sigma) + \pi\theta(\tau - \sigma), r(\sigma)] + \qquad (3.55)$$

$$+ e^{ip_0\pi} e^{ir(\tau)} \Psi[l(\sigma), r(\sigma) - \pi\theta(\tau - \sigma)] \Big]$$

There is no difficulty in separating the action of Q_f into left and right parts because of the condition $f_+(\pi/2) = 0$. We can rewrite (3.55) as

$$Q_f \Psi \Rightarrow \hat{Q}_f \hat{\Psi} + e^{i\pi(p_0 + 1/2)} \hat{\Psi} \hat{Q}_f = \hat{Q}_f \hat{\Psi} - (-1)^{G_\Psi} \hat{\Psi} \hat{Q}_f \,, \qquad (3.56)$$

where

$$\hat{Q}_f = \int_0^{\pi/2} d\tau f_+(\tau) K e^{i\frac{\pi}{4}} e^{i\hat{l}(\tau)} e^{i\pi \int_0^\tau d\rho\, \hat{p}(\rho)} \,. \qquad (3.57)$$

The operator $\hat{Q}_f$ commutes with the midpoint insertion $e^{3i\bar{\phi}/2}$,

$$\hat{Q}_f M = M \hat{Q}_f \,. \qquad (3.58)$$

Again, this follows immediately from the fact that we have restricted attention to operators with $f_+(\pi/2) = 0$. If we did not make this restriction, the derivative $i\hat{p}(\pi/2)$ would appear in (3.57), and when commuting with M might combine with the infinite factor K to give a nonzero commutator instead of (3.58).

Now let us consider conditions (a-c) above. In each case we use a BRST operator Q_f given by (3.50) with $f_+(\sigma)$ satisfying (3.51), (3.52).

(a) $Q_f^2 \Psi = 0$. We have

$$Q_f^2 \Psi[\phi(\sigma)] = \int_0^\pi d\tau \int_0^\pi d\rho\, f_+(\tau) f_+(\rho) K^2 e^{i\pi/2} e^{i\pi\theta(\tau-\rho)} e^{i\phi(\tau) + i\phi(\rho)} \times$$

$$\times\, \Psi\left[\phi(\sigma) + \pi\left(\theta(\tau - \sigma) + \theta(\rho - \sigma)\right)\right] . \qquad (3.59)$$

The factor $e^{i\pi\theta(\tau-\rho)}$ is explicitly odd under $\rho \leftrightarrow \tau$ except at $\rho = \tau$, and the rest of the integrand is even under this exchange, so that naively $Q_f^2 = 0$. One might worry that, since the integrand contains the divergent factor K^2 and does not vanish at $\rho = \tau$, a nonvanishing result might nonetheless emerge. In appendix D we show explicitly that this is not the case when Q_f^2 acts on a well-behaved functional, such as those that lie in the ghost Fock space. For these states, each factor of K cancels a corresponding factor of $1/K$ when the shifted functional is expressed as a well-behaved functional (analogous to the calculation in (3.44)), and the regulated Q_f is indeed nilpotent as $x \to 1$. The question of what constitutes a well-behaved functional is discussed at the end of this section.

We can repeat this analysis in split string language, using the operator representation of Q_f developed above. An identical argument shows that

$$\hat{Q}_f^2 \hat{\Psi} = 0\,, \tag{3.60}$$

when $\hat{Q}_f^2$ acts on a well-behaved operator. Consequently, using (3.56),

$$\begin{aligned} Q_f^2 \Psi \Rightarrow\ & \hat{Q}_f[\hat{Q}_f\hat{\Psi} - (-1)^{G_\Psi}\hat{\Psi}\hat{Q}_f] + (-1)^{G_\Psi}[\hat{Q}_f\hat{\Psi} - (-1)^{G_\Psi}\hat{\Psi}\hat{Q}_f]\hat{Q}_f \\ =\ & -(-1)^{G_\Psi}\hat{Q}_f\hat{\Psi}\hat{Q}_f + (-1)^{G_\Psi}\hat{Q}_f\hat{\Psi}\hat{Q}_f \\ =\ & 0\,, \end{aligned} \tag{3.61}$$

where we have used the fact that $\hat{Q}_f$ contains a factor of M and thus has ghost number one.

While we have gone through this demonstration in some detail to be clear about the formalism, the result that $Q_f^2 = 0$ when acting on a well-behaved state essentially follows from the anticommutator (3.14). In deriving this property of Q_f we have not used either of the two conditions (3.51), (3.52) on the weight function f_+.

(b) $\int Q_f \Psi = 0$. We have

$$\begin{aligned} \int Q_f\Psi[\phi(\sigma)] = \int \mathcal{D}\phi \int_0^\pi d\tau\, e^{-\frac{3i}{2}\phi(\frac{\pi}{2})} \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta\left(\phi(\sigma) - \phi(\pi-\sigma)\right) \times \\ \times\, f_+(\tau) K e^{i\pi/4} e^{i\phi(\tau)} \Psi[\phi(\sigma) + \pi\theta(\tau-\sigma)] \end{aligned} \tag{3.62}$$

Performing a change of variables $\phi(\sigma) \to \phi(\sigma) - \pi\theta(\tau-\sigma)$, this becomes

$$\begin{aligned} \int \mathcal{D}\phi \int_0^\pi d\tau\, e^{-\frac{3i}{2}\left(\phi(\frac{\pi}{2}) - \pi\theta(\tau-\frac{\pi}{2})\right)} f_+(\tau) K e^{-i\pi/4} e^{i\phi(\tau)} \times \\ \times \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta\left(\phi(\sigma) - \pi\theta(\tau-\sigma) - \phi(\pi-\sigma) + \pi\theta(\tau-\pi+\sigma)\right) \Psi[\phi(\sigma)] \end{aligned} \tag{3.63}$$

The reason that this integral vanishes is that $f_+(\tau)$ is symmetric under $\tau \to \pi - \tau$, as is the overlap delta function, since

$$\begin{aligned} \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta\left(\phi(\sigma) - \pi\theta(\tau-\sigma) - \phi(\pi-\sigma) + \pi\theta(\tau+\sigma-\pi)\right) = \\ = \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta\left(\phi(\sigma) - \pi\theta(\pi-\tau-\sigma) - \phi(\pi-\sigma) + \pi\theta(\sigma-\tau)\right) \end{aligned} \tag{3.64}$$

while the phase factor, $\exp i[\phi(\tau) + \frac{3\pi}{2}\theta(\tau - \frac{\pi}{2})]$, is odd. To see this note that (for $\tau < \pi/2$) the delta function enforces:

$$\phi(\tau) = \phi(\pi-\tau) - \pi\theta(\tau+\tau-\pi) + \pi\theta(0) = \phi(\pi-\tau) + \frac{\pi}{2}\,, \tag{3.65}$$

since, as discussed above, $\theta(0) = 1/2$. Consequently (for $\tau < \pi/2$)

$$\phi(\tau) + \frac{3\pi}{2}\theta\left(\tau - \frac{\pi}{2}\right) = \phi(\tau) = \phi(\pi - \tau) + \frac{\pi}{2} = \phi(\pi - \tau) + \frac{3\pi}{2}\theta\left(\frac{\pi}{2} - \tau\right) - \pi\,. \quad (3.66)$$

We again need not worry about the infinite factor K for well-behaved functionals. In this case, this is because the weight function $f_+(\tau)$ vanishes at the point $\tau = \pi/2$, which is the one point where the phase factor fails to be odd under $\tau \to \pi - \tau$. If we had chosen, for example, $f_+(\tau) = f_-(\tau) = 1$ we would find that the resulting operator $Q_f = c_0$ would not satisfy $\int Q_f\Psi = 0$ even for well-behaved functionals. For example, this integral does not vanish when Ψ is the functional corresponding to the Fock space state $a_1^\dagger|p_0 = 1/2\rangle$.

We can repeat this analysis in the split string operator language. We have

$$\int Q_f\Psi = \mathrm{Tr}\left[M^{-1}\left(\hat{Q}_f\hat{\Psi} - (-1)^{G_\Psi}\hat{\Psi}\hat{Q}_f\right)\right]. \quad (3.67)$$

The integral $\int Q_f\Psi$ vanishes unless the ghost number of Ψ equals two, and since

$$[\hat{Q}_f, M] = [\hat{\Psi}, M] = 0\,,$$

we see that (3.67) vanishes by cyclicity of the trace. In general we must be careful when using the cyclicity property of the trace, since generic operators cannot be commuted inside the trace. For operators $\hat{\Psi}$ associated with well-behaved states this operation is valid, however.

We have thus seen that a general ghost BRST operator of the type we are interested in satisfies $\int Q\Psi = 0$ for well-behaved states Ψ. This result, however, depends crucially on the vanishing of the weight function $f_+(\pi/2)$ at the midpoint, which simplifies the split string formalism.

(c) $Q_f(A \star B) = (Q_fA) \star B + (-1)^{G_A}A \star (Q_fB)$**.** We use the operator formalism developed above, where

$$Q_f(A) = \hat{Q}_f\hat{A} - (-1)^{G_\Psi}\hat{A}\hat{Q}_f\,, \quad (3.68)$$

Thus for

$$A \star B = \hat{A}M\hat{B} \quad (3.69)$$

we have

$$Q_f(A \star B) = \hat{Q}_f\hat{A}M\hat{B} - (-1)^{G_A+G_B}\hat{A}M\hat{B}\hat{Q}_f\,. \quad (3.70)$$

On the other hand,

$$\begin{aligned} Q_f(A) \star B + (-1)^{G_A}A \star Q_f(B) = &\left(\hat{Q}_f\hat{A} - (-1)^{G_A}\hat{A}\hat{Q}_f\right)M\hat{B} + \\ &+(-1)^{G_A}\hat{A}M\left(\hat{Q}_f\hat{B} - (-1)^{G_B}\hat{B}\hat{Q}_f\right). \end{aligned} \quad (3.71)$$

Using the fact (3.58) that $\hat{Q}_f$ commutes with M we see that (3.70) and (3.71) are indeed equal.

While the derivation property of Q_f follows fairly immediately in the split string formalism, we should emphasize that this formalism, and therefore our demonstration of this result, depended on the vanishing midpoint condition $f_+(\pi/2) = 0$ in the definition of Q.

In this subsection we have shown that a general class of pure ghost BRST operators satisfy the axioms of cubic string field theory when acting on states in the bosonized ghost Fock space. We have stated that these results follow when the states acted on are "sufficiently well-behaved", but we have not given a precise description of what this criterion entails. Making this statement precise amounts to giving a precise algebraic formulation of string field theory, which is an important outstanding problem whose resolution promises to shed light on many questions about the theory. In the arguments in this section we have actually used fairly weak properties of the states Ψ. Although we have only made these results precise in the case of states living in the bosonized ghost Fock space, for the class of BRST operators we have considered here these statements should hold for any functional $\Psi[\phi_n]$ which depends on ϕ_n for large n in the same way as the Fock space vacuum, that is as $\Psi \sim \exp[-n\phi_n^2/2]$. More precisely, we need to have $\Psi \sim \exp[-\sum_{n,m} \phi_n P_{nm} \phi_m/2]$, where the eigenvalues of P_{nm} go as $n(1 + O((1/n)^\epsilon))$ for large n and some suitable value of ϵ. When working with split string operators $\hat{\Psi}$ it seems that a slightly different criterion is needed for a functional to be well-behaved. Namely, the functional $\Psi[l; r]$ should behave as $\exp[-(2k+1)l_{2k+1}^2/2]$ in both the right and left degrees of freedom. An important problem in developing the split string field theory formalism further is the determination of precisely how the criteria for a string state and the corresponding string operator are related. Some progress in understanding the nature of well-behaved operators in the related context of noncommutative geometry was made in [37].

While the subtleties of precisely which string fields are allowed in the algebraic formulation of string field theory will probably become crucial in analyzing the theory around the D25-brane, which has a BRST operator Q_B which couples the matter and ghost sectors, it seems that the most important property we have used here to prove (a–c) for pure ghost operators Q is the vanishing of the weight function $f_\pm(\pi/2) = 0$ at the midpoint. This allows the BRST operator to be split in a straightforward way into right and left parts, so that the axioms of cubic string field theory follow in a fairly straightforward fashion. As we discuss in section 5, this gives the vacuum string field theory proposed by RSZ a much simpler structure than the string field theory with BRST operator Q_B.

4. Gauge invariance and physical observables

The cubic open string field theory is invariant under the gauge transformation

$$\Psi \to \Psi + \delta\Psi\,; \qquad \delta\Psi = Q(\Lambda) + \Psi \star \Lambda - \Lambda \star \Psi\,, \tag{4.1}$$

where Λ is a string field of ghost number 0 (ghost momentum $p_0 = -3/2$). In the split string operator formalism this variation takes the form

$$\hat{\Psi} \to \hat{\Psi} + \delta\hat{\Psi}\,; \qquad \delta\hat{\Psi} = \hat{Q}_f\hat{\Lambda} - \hat{\Lambda}\hat{Q}_f + \hat{\Psi}M\hat{\Lambda} - \hat{\Lambda}M\hat{\Psi}\,. \tag{4.2}$$

We can write this variation in terms of a *covariant derivative*, $\hat{\mathcal{D}}_\Psi$, which is a functional of the string field $\hat{\Psi}$. Let us define the covariant derivative, $\hat{\mathcal{D}}_\Psi$, of a string functional Φ of ghost number zero (in the *fundamental* representation) to be

$$\hat{\mathcal{D}}_\Psi \equiv \hat{Q}_f + \hat{\Psi}M\,; \qquad \hat{\mathcal{D}}_\Psi\hat{\Phi} = \hat{Q}_f\hat{\Phi} + \hat{\Psi}M\hat{\Phi}\,. \tag{4.3}$$

As discussed in section 3, $M = \exp[3i\phi(\pi/2)]$ goes through the other operators. Thus, under a gauge transformation

$$\hat{\Psi} \to \hat{\Psi} + \delta\hat{\Psi}\,; \qquad \delta\hat{\Psi} = \hat{\mathcal{D}}_\Psi\hat{\Lambda} - \hat{\Lambda}\hat{\mathcal{D}}_\Psi = \left[\hat{\mathcal{D}}_\Psi, \hat{\Lambda}\right]. \tag{4.4}$$

Define the field strength, F, to be

$$F = -\frac{\partial S}{\partial \Psi} = Q(\Psi) + \Psi \star \Psi; \qquad \hat{F} = \hat{Q}_f\hat{\Psi} + \hat{\Psi}\hat{Q}_f + \hat{\Psi}M\hat{\Psi}\,. \tag{4.5}$$

Then it follows that under a gauge transformation,

$$\delta\hat{F} = -\hat{\Lambda}M\hat{F} + \hat{F}M\hat{\Lambda} = -\left[\hat{\Lambda}M, \hat{F}\right]. \tag{4.6}$$

Clearly, if the equation of motion, $\hat{F} = 0$, is satisfied for $\hat{\Psi}$, it will be satisfied for $\hat{\Psi} + \delta\hat{\Psi}$.

Note that

$$\hat{F} = M^{-1}\hat{\mathcal{D}}_\Psi\hat{\mathcal{D}}_\Psi = \frac{1}{2}M^{-1}\{\hat{\mathcal{D}}_\Psi, \hat{\mathcal{D}}_\Psi\}\,, \tag{4.7}$$

The reason that the anticommutator of the covariant derivative arises, as opposed to the commutator, is because $\hat{\mathcal{D}}_\Psi$ has ghost number one. Also, if Ψ is a solution of the equation of motion, then $\hat{\mathcal{D}}_\Psi^2 = 0$, and $\hat{\mathcal{D}}_\Psi$ is the BRST operator in the new vacuum defined by shifting the string field by Ψ.

We are interested in the largest class of gauge invariant observables. We believe that the most general physical, gauge invariant observables of open string field theory are arbitrary functions of the eigenvalues of $\hat{\mathcal{D}}_\Psi$. Under a gauge transformation $\hat{\mathcal{D}}_\Psi$ changes by

$$\hat{\mathcal{D}}_\Psi \to \hat{\mathcal{D}}_{\Psi+\delta\Psi} = \hat{\mathcal{D}}_\Psi - \left[\hat{\Lambda}, \hat{\mathcal{D}}_\Psi\right] \tag{4.8}$$

and thus its eigenvalues are gauge invariant functionals of the string field Ψ.

What remains as a challenge is to figure out which functions of these eigenvalues are particularly interesting observables, with interesting physical content. There are many such eigenvalues and one can construct many functions of these. In this theory, unlike ordinary gauge theory, it is not at all obvious how to pass from the eigenvalues of the covariant derivative to Wilson loop observables. In ordinary gauge theory the covariant derivative carries a space-time label and one can use it to construct local gauge invariant observables such as $\mathrm{Tr}[\mathcal{D}_\mu, \mathcal{D}_\nu]^2$. A harder case is that of non-commutative field theory. Consider two-dimensional noncommutative gauge theory. In the operator formulation, as discussed in [38], the theory is formulated in terms of covariant derivative operators, $\mathcal{D}_\mu, (\mu = 1, 2)$ in Hilbert space. Although spatial labels are now absent, one can construct the noncommutative analogoue of Wilson loops by taking $\mathrm{Tr} \prod_i \exp[\lambda_i^\mu \mathcal{D}_\mu]$, along a polygon generated by $\{\lambda_i^\mu\}$. In the string field theory context, we have neither space-time labels or space time indices for our covariant derivative, and it is not obvious how to construct interesting or useful observables. We would like to study something like $\langle \mathrm{Tr}\, f_1(\hat{\mathcal{D}}_\Psi)\, \mathrm{Tr}\, f_2(\hat{\mathcal{D}}_\Psi)\rangle$, for appropriate functions f_1 and f_2, with the hope of using these to probe the spectrum of the theory and to construct scattering amplitudes. The problem is then to find the appropriate class of f_i's.

5. Finding ghost solutions

We now have a complete description in the split string operator language of a class of cubic string field theories with pure ghost BRST operators Q. These theories correspond to the vacuum string field theories proposed in [24]. We are interested in finding solutions to the string field theory equations of motion

$$Q\Psi + \Psi \star \Psi = 0\,. \tag{5.1}$$

In 5.1 we discuss ghost number zero projection operators in the ghost sector. In 5.2 we describe solutions of (5.1) using a pure ghost BRST operator of the type described in 3.3. We find a general class of solutions, but find that the action vanishes generically in the split string formalism for any solution of (5.1). We discuss possible interpretations of this result. In 5.3 we discuss related solutions of Witten's original string field theory on the D25-brane, which has a more complicated BRST operator Q_B that couples the matter and ghost sectors.

5.1 Ghost projectors

A first step in constructing a solution to the string field theory equation (5.1) in the ghost sector is to find a solution of the projection equation

$$\chi = \chi \star \chi \tag{5.2}$$

in the ghost sector. Solutions to this equation must have ghost number zero. In the bosonized formulation, such solutions are of the form

$$\hat{\chi} = e^{-3i\bar{\phi}/2}\hat{P} \tag{5.3}$$

where $\hat{P} = |\eta\rangle\langle\eta|$ is a projection operator on the space of midpoint-independent half-string states. As discussed in in [22, 14], an example of a rank one projection operator of this form is given by the zero-momentum bosonic sliver state constructed in [28, 29]. This state is closely related to the D-instanton sliver discussed in section 2. It is useful to be slightly more explicit about this construction, and to compare the bosonized and fermionic representations of the ghost projector.

It was shown in [29] that the sliver state can be described in terms of Virasoro operators acting on the vacuum through

$$|\Xi\rangle = \exp\left(\sum_{n=1}^{\infty} \alpha_{2n} L_{-2n}\right)|0\rangle = \exp\left(-\frac{1}{3}L_{-2} + \frac{1}{30}L_{-4} + \cdots\right)|0\rangle \tag{5.4}$$

where α_{2n} are a calculable series of constants. This formula is valid both in the ghost and matter sectors of the theory. The full sliver defined in conformal field theory is given by the tensor product of the ghost and matter slivers; this state is discussed in detail in [39]. In the matter sector, the state (5.4) is defined using the zero-momentum vacuum state $|0\rangle$ and the usual matter Virasoro generators

$$\begin{aligned} L_n^{(\text{matt})} &= \frac{1}{2}\sum_{m\neq n}\sqrt{|m(n-m)|}\; a_m a_{n-m} + \sqrt{n}\; a_n p_0 \qquad n \neq 0 \\ L_0^{(\text{matt})} &= \frac{1}{2}p_0^2 + \sum_{m=1}^{\infty} m\; a_{-m} a_m\,. \end{aligned} \tag{5.5}$$

In fermionic language, the ghost sliver is defined through (5.4) where the vacuum is taken to be the ghost number zero vacuum $|\Omega\rangle = b_{-1}|-\rangle$, and the Virasoro generators are given by

$$L_n^{(\text{g})} = (n-m) b_{n+m} c_{-m}\,. \tag{5.6}$$

To construct the ghost sliver in the bosonized language, we use (5.4) where the ground state is given by

$$|\Omega\rangle \rightarrow |p_0 = -3/2\rangle \tag{5.7}$$

and the Virasoro generators are given by [35, 30]

$$\begin{aligned} L_n^{(\phi)} &= \frac{1}{2}\sum_{m\neq n}\sqrt{|m(n-m)|}\; a_m a_{n-m} + \sqrt{n}\; a_n\left(p_0 - \frac{3}{2}n\right), \qquad n \neq 0 \\ L_0^{(\phi)} &= \frac{1}{2}p_0^2 + \sum_{m=1}^{\infty} m\, a_{-m} a_m - \frac{1}{8}\,. \end{aligned} \tag{5.8}$$

These bosonized ghost Virasoro generators can be related to (5.6) using the bosonization formulae from section 3.1. Thus, the fermionic and bosonized forms of the ghost sliver defined through (5.4) are seen to be equivalent. In the bosonized language, this state is indeed a projector of the form (5.3), as desired. Once we have a single projector of this form, a variety of other projectors can be constructed which are also of the form (5.3), with $\hat{P}$ a rank one projector on the space of midpoint-independent half-string states, using the methods developed in [22, 14].

5.2 Solutions for pure ghost Q

Now that we have found a class of ghost number zero states satisfying the projection equation (5.3), we wish to use this state to construct a solution to the full ghost equation (5.1). A ghost projector (5.3) can be written in operator form as

$$\hat{\chi} = |\eta\rangle e^{-3i\bar{\phi}/2}\langle\eta| \tag{5.9}$$

where $|\eta\rangle$ is a half-string state such that $\hat{\chi}$ satisfies (5.2). Let us consider the state associated with the operator

$$\hat{\Psi} = -2\left(\hat{Q}\hat{\chi} + \hat{\chi}\hat{Q}\right). \tag{5.10}$$

This does not correspond to the exact state $Q\chi$, since such an exact state corresponds to the operator

$$\hat{Q}\hat{\chi} - \hat{\chi}\hat{Q} \tag{5.11}$$

when $\hat{\chi}$ has ghost number zero. The equation (5.1) for the state (5.10) reads (using $\hat{Q}^2 = 0$)

$$\hat{Q}\hat{\chi}\hat{Q} = \hat{Q}\hat{\chi}M\hat{Q}\hat{\chi} + \hat{\chi}\hat{Q}M\hat{\chi}\hat{Q} + \hat{Q}\hat{\chi}M\hat{\chi}\hat{Q}\,, \tag{5.12}$$

where $M = e^{\frac{3i}{2}\bar{\phi}}$ commutes with $\hat{Q}$. This equation will be satisfied if

$$\hat{\chi}M\hat{\chi} = \hat{\chi}\,; \qquad \hat{\chi}M\hat{Q}\hat{\chi} = \hat{\chi}\hat{Q}M\hat{\chi} = 0\,. \tag{5.13}$$

Thus, if (5.9) is a rank one projector onto a midpoint-independent half-string state $|\eta\rangle$ with

$$\langle\eta|\eta\rangle = 1 \tag{5.14}$$

and

$$\langle\eta|\hat{Q}|\eta\rangle = 0 \tag{5.15}$$

then we have constructed a solution to (5.1). It is easy to see that such solutions exist for generic Q. Indeed, choosing any rank one projection, such as the state described in the previous subsection, we can calculate

$$\begin{aligned}\alpha_1 &= \langle\eta|\hat{Q}_1|\eta\rangle\,,\\ \alpha_2 &= \langle\eta|\hat{Q}_2|\eta\rangle\end{aligned} \tag{5.16}$$

for any pair of pure ghost BRST operators Q_1, Q_2. If both $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ then we can construct a new BRST operator

$$Q' = \alpha_2 Q_1 - \alpha_1 Q_2 \tag{5.17}$$

which satisfies

$$\langle\eta|\hat{Q}'|\eta\rangle = 0\,. \tag{5.18}$$

Thus, the expression (5.10) gives us a construction of a wide class of solutions to (5.1) for various choices of the pure ghost BRST operator Q and the ghost projector (5.3). Combining this with a matter projector should then give a solution to the string field theory equations of motion in the full cubic string field theory when the BRST operator Q is of the form (3.50). According to the philosophy expressed in [27, 22, 14], this allows us to construct an arbitrary configuration of multiple Dp-branes in the vacuum string field theory postulated in [24]. Unfortunately, however, there is a problem with this interpretation. Computing the action of a state of the form

$$\Psi = \Psi_m \otimes \Psi_g\,, \tag{5.19}$$

where Ψ_m is a matter projector such as (2.34) and Ψ_g is a ghost solution of the form (5.10), we find that

$$S = -\frac{1}{6}\int \Psi \star Q\Psi = -\frac{4}{3}\,\mathrm{Tr}\,\hat{Q}\hat{\chi}\hat{Q}\hat{\chi}\hat{Q} = 0\,. \tag{5.20}$$

Thus, our solutions seem to have vanishing action.[1]

This problem is actually generic to solutions in the split string formalism. Assume that the BRST operator Q can be represented in split string language by an operator $\hat{Q}$ so that

$$Q\Psi \Rightarrow \hat{Q}\hat{\Psi} - (-1)^{G_\Psi}\hat{\Psi}\hat{Q} \tag{5.21}$$

where $\hat{Q}$ has the properties we derived for operators of the form (3.50) in section 3, namely $\hat{Q}^2 = 0$ and $\mathrm{Tr}\,\hat{Q}\hat{\Psi} = \mathrm{Tr}\,\hat{\Psi}\hat{Q}$. Then for any solution of the equation

$$Q\Psi = \Psi \star \Psi \tag{5.22}$$

we have

$$\begin{aligned}
\int \Psi \star Q\Psi &\Rightarrow \mathrm{Tr}\,\hat{\Psi}(\hat{Q}\hat{\Psi} + \hat{\Psi}\hat{Q}) \\
&= 2\,\mathrm{Tr}\,\hat{Q}\hat{\Psi}\hat{\Psi} \\
&= -2\,\mathrm{Tr}\,M^{-1}\hat{Q}(\hat{Q}\hat{\Psi} + \hat{\Psi}\hat{Q}) \\
&= -2\,\mathrm{Tr}\,M^{-1}\hat{Q}\hat{\Psi}\hat{Q} \\
&= 0\,.
\end{aligned} \tag{5.23}$$

[1]Note added: in addition to the fact that the action vanishes for these solutions, which as discussed in the text is generic in the split string formalism, these particular solutions also have trivial BRST cohomology, so they cannot be interpreted as the ghost part of a Dp-brane. The vanishing of the cohomology can be easily seen by noting that these solutions give a BRST operator $\tilde{Q} = (1-2\hat{\chi})\hat{Q}(1-2\hat{\chi})$. Such a BRST operator is gauge equivalent under a complex gauge transformation $U = \exp[i\pi\hat{\chi}]$ to $\hat{Q}$. We would like to thank I. Ellwood for discussions on this point.

In this calculation we have only used the above-mentioned properties of the BRST operator Q and the cyclicity of the trace. All these properties were explicitly verified in section 3 for the simple class of pure ghost BRST operators which are linear in c with a weight which vanishes at the midpoint $\sigma = \pi/2$, when dealing with string fields Ψ which are in the string Hilbert space, or which are similarly well-behaved.

There are a number of possible explanations for the vanishing of the action (5.23). We list here some of the most likely possibilities:

(A) One possibility is that the solution of the vacuum string field theory corresponding to a Dp-brane lies outside the Hilbert space. A crucial question, to which we do not yet know the answer, is precisely what the space of allowable string field functionals should be. It seems that this space is larger than the string Hilbert space. Examples of states which probably lie outside the Hilbert space are given by the matter sliver state discussed in appendix B and the matter state $|0\rangle \star |0\rangle$. The sliver state lies outside the Hilbert space of normalizable Fock space states when the matrix S given in (B.4) has an eigenvalue of ± 1. There is some evidence [39, 40] that indeed S has an eigenvalue of -1, taking the sliver state just outside the Hilbert space. This in turn suggests that the matrix V from (A.14) has an eigenvalue of -1 in the odd-odd sector, so that the state $|0\rangle \star |0\rangle$ also lies just outside the normalizable Fock space. In order to have a solution of (5.1) which has nonvanishing action in the cubic string field theory with a pure ghost BRST operator of the form (3.50), it may be that we need to understand how the properties of Q derived in 3.3 need to be modified for states in a larger class of functionals. We must be careful, however, not to expand the space of allowed functionals too far, or the structure of the theory, which depends critically on the properties of Q, will break down. One piece of evidence which may support this possible explanation of (5.23) is the recent numerical calculation of a ghost solution with nonvanishing action in a level truncation approximation around the ghost sliver in the vacuum string field theory [41]. At least to level 2 truncation, this solution appears to be numerically stable, and has a structure roughly compatible with the ansatz (5.10). This may suggest that the action does not actually vanish for all solutions, and that the solutions of interest indeed lie just far enough outside the Fock space that the steps used in (5.23) are not exactly valid, but that the solution is still a well-enough behaved functional that the structure of the string field theory still holds.

(B) Another possible explanation for the vanishing of the action (5.23) for a generic solution of the theory with a pure ghost BRST operator Q is that the vacuum string field theory postulated in [24] may have an action which is related by an infinite overall normalization factor to the action of Witten's original theory in the condensed tachyon vacuum. Since there are no perturbative physical states in the stable vacuum [13, 25], there is no simple way of calculating the overall normalization of the action in the vacuum string field theory other than by comparing the energy of soliton configurations such as those considered here. We describe in appendix B a set of normalization factors which are needed to convert from the split string formalism to the usual Fock space conventions, in the matter sector. If as mentioned above, the matrix V has an eigenvalue -1 in the odd-odd block, the product of conversion factors $\gamma_1\gamma_3$ vanishes. It may be that similar formally vanishing or infinite normalization factors are needed to establish the finite energy of a Dp-brane solution in the vacuum string field theory.

(C) A final possible explanation for the vanishing of (5.23) may be that the vacuum string field theory hypothesis, in which the BRST operator Q is taken to be linear in the ghost field and vanishing at the string midpoint, may simplify the theory too much. Indeed, the derivation of the properties of the BRST operator Q, such as $\int Q\Psi = 0$, in section 3 depended crucially on the vanishing of the weight function $f(\sigma)$ at the string midpoint $\sigma = \pi/2$. In the original string field theory of Witten with the BRST operator Q_B coupling the matter and ghost sectors, the condition that $\int Q_B\Psi = 0$ depends on anomaly cancellation between the matter and ghost sectors, which occurs only in dimension 26 [31]. It may be that by choosing a particularly simple class of BRST operators for which the matter and ghost sectors decouple and for which the anomaly plays less of a role, we have lost some important structure in the theory, so that the action will indeed generically vanish for solutions of the string field theory equations. It seems likely that the theory in this case retains something like the topological structure of the original theory (for example the existence of D-brane solutions), but perhaps not all the dynamical details. One issue which may be related to this possibility is the question of how to find poles at nonzero p^2 in the theory around the Dp-brane solitons in the vacuum string field theory. Some discussion of questions related to this issue was given in [39], but it is still unclear how the physical spectrum of the usual open string can be constructed using a Q with no dependence on the matter sector.

We conclude this discussion of solutions of the equations of motion in the case of pure ghost BRST operators with a few remarks on how these solutions might be checked using level truncation. The split string operator $\hat{Q}$ given in (3.57) can be described in terms of operators Q_l, Q_r acting on the space of full string functionals through

$$\begin{aligned} Q_l\Psi &\Rightarrow \hat{Q}\hat{\Psi} \\ Q_r\Psi &\Rightarrow \hat{\Psi}\hat{Q}\,. \end{aligned} \tag{5.24}$$

From the definition (3.50), we have

$$\begin{aligned} Q_l &= \frac{1}{2\pi}\int_0^{\pi/2} d\sigma\left(f_+(\sigma)c^+(\sigma) + f_-(\sigma)c^-(\sigma)\right) \\ Q_r &= \frac{1}{2\pi}\int_{\pi/2}^{\pi} d\sigma\left(f_+(\sigma)c^+(\sigma) + f_-(\sigma)c^-(\sigma)\right). \end{aligned} \tag{5.25}$$

This gives us a way of explicitly defining Q_l, Q_r in terms of fermionic ghost raising and lowering operators. For example, when $f_+ = f_- = 1$, we have

$$Q = c_0\,, \tag{5.26}$$

and

$$\begin{aligned} Q_l &= \frac{1}{2}c_0 + \sum_k \frac{(-1)^k}{\pi(2k+1)}c_{2k+1} \\ Q_r &= \frac{1}{2}c_0 - \sum_k \frac{(-1)^k}{\pi(2k+1)}c_{2k+1}\,. \end{aligned} \tag{5.27}$$

In this language the solution (5.10) is given by

$$|\Phi\rangle = 2(Q_r - Q_l)|\chi\rangle\,, \tag{5.28}$$

where $|\chi\rangle$ is the ghost Fock space state associated with a projection operator satisfying $|\chi\rangle \star |\chi\rangle = |\chi\rangle$. This choice of Q does not satisfy all the desired axioms, since $f(\pi/2) \neq 0$, but the same approach gives the analogous operators Q_l, Q_r for any other choice of f. It would be interesting to check using level-truncation whether states of the form (5.28) indeed correspond with solutions of the vacuum string field theory equations of motion having nonzero action. Some preliminary numerical work towards finding ghost solutions appeared in [41]; the results of this work seem very roughly compatible with this solution ansatz (for example, the two states considered there with a single c acting on the sliver are weighted much more heavily than the single state with a bcc product acting on the sliver in a 3-state truncation), although more data is needed to make any definitive statement about this question.

5.3 Solutions of the coupled matter-ghost theory

It is clearly important to understand how the approach we have taken in this work can be generalized to the perturbative string field theory around a D25-brane given by Witten's original formulation of the theory with BRST operator Q_B. All of the formal structure we have developed here for describing the string star product $\star$ and integral $\int$ in the matter and ghost sectors applies equally well to the D25-brane theory. The difficult remaining technical problem is to formulate the split string description of Q_B and to check that it satisfies the desired axioms. We leave the details of this formulation to future work, but make some general comments here outlining some possible issues which may arise in solving this problem.

The BRST operator Q_B is given by

$$Q_B = \int_0^\pi d\sigma \left[c_+(\sigma) \left(T^{(m)}_{--}(\sigma) + \frac{1}{2} T^{(g)}_{--}(\sigma) \right) + (+ \leftrightarrow -) \right]. \qquad (5.29)$$

This operator can be rewritten in a fairly straightforward fashion in terms of the bosonized ghosts in a form similar to (3.50), where f is replaced by expressions of the form $\partial x \partial x$ in the matter sector and $\partial\phi\partial\phi + \partial^2\phi$ in the ghost sector. This operator can in principle be split into parts acting on the left and right of the string along the lines of (3.56), (5.25). This splitting is more subtle, however, than the splitting in the case of a pure ghost operator with weight vanishing at the midpoint. The verification of the axioms of string field theory using the split form of Q_B is also much more subtle. For example, as demonstrated in [30], the proof that $\int Q_B \Psi = 0$ depends crucially on anomaly cancellation between the matter and ghost sectors.

Despite these complications, we believe that a split string formulation of the BRST operator Q_B is possible. It may be, however, that the action of Q_B on even a well-behaved state is more complicated than (3.56), due to the midpoint anomaly. If, indeed, the structure of the split BRST operator is more complicated in the case of Q_B than in the pure ghost case, it may be possible to avoid the vanishing action problem of (5.23), even for well-behaved string states. This would fit well with Sen's conjectures [13] and results from level truncation indicating the existence of nontrivial solutions with nonzero action [42, 43, 44].

It may even be that the solution corresponding to the locally stable vacuum in the original D25-brane string field theory may have something like the form of (5.10), and that subtleties related to the midpoint anomaly may be responsible for giving this solution a nonvanishing action. It is worth pointing out that the solution (5.10) is closely related to a solution $Q_l|I\rangle$ which was considered some time ago in [16], and which leads to a purely cubic string field theory action. Indeed, (5.10) is equivalent to this solution when the rank one projection $\hat{\chi}$ is replaced by the infinite rank projection $\hat{I}$. It may be that the solutions (5.10) with $\hat{\chi}$ a rank one projector correspond to single D-branes, while the solution with $\hat{\chi} = I$ may represent a $U(\infty)$ theory of an infinite number of space-filling D25-branes, such as is useful in K-theory constructions [45]. In any case, we believe that the technology we have developed here provides a good basis for further development towards finding an analytic solution of the full string field theory with BRST operator Q_B, as well as a simple formulation of this theory using the split string approach.

6. Conclusions

In this paper we have continued the work begun in [14] of developing a complete description of string field theory in terms of an algebra of operators on half-string states. In [14] we developed this formalism in the matter sector of the theory and

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identified certain states of interest as projection operators in the matter sector. In this paper we have developed the split string formalism for the string field star product and integral in the ghost sector, using the bosonized description of the ghosts as a single bosonic scalar field. We also described a class of pure ghost BRST operators Q in the split string language. These operators were conjectured in [24] to define a cubic string field theory in the closed string vacuum which is formally a field theory of open strings but which has no physical open string states. This vacuum string field theory shares with the earlier proposal of a purely cubic string field theory [16] the virtue of being independent of a choice of conformal field theory background [39].

We used the split string formalism for the ghost sector to find a class of solutions to the string field theory equations of motion. We showed that these solutions all have vanishing action, and that in fact the split string formalism for the pure ghost Q's considered here gives vanishing action for any solution which is described by a well-behaved functional, such as a state in the Hilbert space. We speculated that this generically vanishing action might be reconciled with the expectation that D-brane states have finite nonzero action in several ways. It may be that the solutions of interest lie far enough outside the usual string Hilbert space that the arguments used here do not apply. It may be that the action of the vacuum string field theory is related to that of the usual cubic string field theory with BRST operator Q_B by a formally infinite multiplicative factor, leading to a vanishing action for D-branes in the vacuum string field theory. It may also be that the simplifications arising from the assumption that the BRST operator have a pure ghost form and have vanishing action at the string midpoint leads to a theory with less structure than the original theory with BRST operator Q_B, so that in the theory with pure ghost Q D-brane solutions still exist but have vanishing action. One example of such a simplification is the absence of anomalies associated with the action of pure ghost Q's at the midpoint, suggesting that the classical vacuum string field theory of RSZ can be consistently defined in any dimension, while the theory with BRST operator Q_B only works in dimension 26.

In this paper we also discussed how the proposed solutions to the ghost equations described here could be studied in level truncation, and we used the solutions in the case of pure ghost Q to suggest a form for solutions to the original D25-brane theory with BRST operator Q_B. This class of solutions is related to the solution suggested in [16] which takes the original theory to a purely cubic action, but in our solutions the string identity operator is replaced by a finite rank projector in the matter and ghost sectors.

The work we have presented here represents only one step towards the complete realization of open string field theory as a background-independent string field theory of operator algebras which will hopefully eventually describe closed strings as well as open strings. There are many major questions which must be answered before this program can be completed. First, we need to find a representation of the usual perturbative string BRST operator Q_B which couples the matter and ghost sectors in terms of operators on the space of half-string states. Next, we need to find explicit analytic solutions of the resulting string field equations of motion, possibly using the form of solution suggested here. After such solutions are found, we need to show that these solutions can be identified with multiple Dp-brane configurations, including the identification of the classical vacuum state which has so far only been understood using level-truncation methods. Once the solution space of the theory is understood, we need to show analytically that in the closed string vacuum there are no perturbative open string excitations and that around the various multiple Dp-brane solutions we have the expected spectrum of open string states.

We are optimistic that the problems just listed represent a series of incremental improvements which can be realized by further development of the formalism described here or by related methods. There are also some larger questions whose solution will probably involve substantial new conceptual developments in the theory. These questions involve the identification of asymptotic closed string states in the open string field theory, the determination of the relevant class of physical observables for the theory, the abstract formulation of the theory as a theory of operator algebras with no reference to any background geometry, and the generalization of all this structure to the superstring. If substantial progress can be made on these problems, open string field theory has the potential to become the first truly background-independent nonperturbative description of string theory.

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A. Relative normalizations using Fock space states and functionals

In this appendix we describe some details of the connection between the split string description of string field theory we are developing here and the more traditional Fock space approach of [30, 31]. In particular there are a number of normalization factors which are different in the two approaches, and we want to be able to translate back and forth between the Fock space and split string languages at will. In this appendix we restrict attention to the matter sector and compare normalization factors between the Fock space and split string pictures. In appendix B we show explicitly how these normalization factors are related in the split string and Fock space descriptions of a particular rank one projection operator considered in [22, 14].

In the Fock space language, the basic elements of cubic string field theory can be defined in terms of three states $\langle I|, \langle V_2|$, and $\langle V_3|$ which lie respectively in the 1-fold, 2-fold, and 3-fold tensor products of the dual string Hilbert space. These three states are calculated in [30], using a particular choice of normalization of the states. We compare those normalizations here to those which arise in the split string approach. To make these comparisons, we define the normalized functional associated with the Fock space vacuum $|0\rangle$ to be

$$\Psi_0 = k \exp\left(-x_0^2 - \frac{n}{2}x_n^2\right), \tag{A.1}$$

where

$$k = \left(\frac{2}{\pi}\right)^{26/4} \prod_{n=1}^{\infty} \left(\frac{n}{\pi}\right)^{26/4}. \tag{A.2}$$

We now compare the normalization factors between the basic string field theory elements $\int$ and $\star$ as defined by (2.16), (2.17) and as defined by $\langle I|, \langle V_2|$ and $\langle V_3|$. The identity state $\langle I|$ is defined so that

$$\int \Psi = \gamma_1 \langle I|\Psi\rangle\,, \tag{A.3}$$

where γ_1 is a numerical constant. In [30], $\langle I|$ is defined through

$$\langle I| = \langle 0| \exp\left[-\frac{1}{2}(a|C|a)\right], \tag{A.4}$$

where

$$C_{nm} = (-1)^n \delta_{nm}\,. \tag{A.5}$$

to calculate the numerical constant γ_1, we compute the integral of the vacuum

$$\begin{aligned} \int \Psi_0 &= \int \prod_{n=0}^{\infty} dx_n \prod_{k=0}^{\infty} \delta(x_{2k+1}) \Psi_0[\{x_n\}] \\ &= (2\pi)^{26/4} \left(\frac{1}{\pi} \cdot \frac{4\pi}{2} \cdot \frac{3}{\pi} \cdot \frac{4\pi}{4} \cdots\right)^{26/4}. \end{aligned} \tag{A.6}$$

Since

$$\langle I|0\rangle = 1\,, \tag{A.7}$$

we have

$$\gamma_1 = (2\pi)^{26/4} \left(\frac{1}{\pi} \cdot \frac{4\pi}{2} \cdot \frac{3}{\pi} \cdot \frac{4\pi}{4} \cdots\right)^{26/4}. \tag{A.8}$$

We will find it useful to leave this expression in the form of an infinite product, although it could be regulated in various ways.

Now let us consider the two-string vertex $\langle V_2|$. This vertex satisfies

$$\int \Psi_1 \star \Psi_2 = \gamma_2 \langle V_2| \left(|\Psi_1\rangle \otimes |\Psi_2\rangle\right) \tag{A.9}$$

for another constant γ_2. $\langle V_2|$ is determined in [30] to be given by

$$\langle V_2| = \left(\langle 0| \otimes \langle 0|\right) \exp\left[-\frac{1}{2}\left(a_{(1)}|C|a_{(2)}\right)\right] \tag{A.10}$$

where $a_{(1,2)}$ are the lowering operators in the first and second Hilbert spaces respectively. It is straightforward to compute

$$\int \Psi_0 \star \Psi_0 = \langle V_2|(|0\rangle \otimes |0\rangle) = 1 \tag{A.11}$$

so that we have

$$\gamma_2 = 1\,. \tag{A.12}$$

We now move on to the three-string vertex $\langle V_3|$. This vertex satisfies

$$\int \Psi_1 \star \Psi_2 \star \Psi_3 = \gamma_3 \langle V_3| \left(|\Psi_1\rangle \otimes |\Psi_2\rangle \otimes |\Psi_3\rangle\right) \tag{A.13}$$

for another constant γ_3. $\langle V_3|$ is determined in [30] to be given by

$$\langle V_3| = \left(\langle 0| \otimes \langle 0| \otimes \langle 0|\right) \exp\left[-\frac{1}{2}\left(\sum_{i,j=1}^{3} a_{(i)}|V^{ij}|a_{(j)}\right)\right] \tag{A.14}$$

for constants V^{ij}_{nm} which are explicitly given in [30, 32, 33, 34]. The three-string vertex can be used to calculate the star product of a pair of states through

$$\langle \Psi_1 \star_F \Psi_2| = \langle V_3| \left(|\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \cdot\right)\,, \tag{A.15}$$

where by $\star_F$ we distinguish the Fock space star product from that defined through (2.8), (2.26) which has a normalization differing by γ_3,

$$\Psi_1 \star \Psi_2 = \gamma_3 \left(\Psi_1 \star_F \Psi_2\right)\,. \tag{A.16}$$

Using (A.15), we can compare

$$\int \Psi_0 \star \Psi_0 = 1 \tag{A.17}$$

with

$$\begin{aligned} \langle I| \left(|0\rangle \star_F |0\rangle\right) &= \langle 0| \exp\left[-\frac{1}{2}\left(a|C|a\right)\right] \exp\left[-\frac{1}{2}\left(a^\dagger|V|a^\dagger\right)\right] |0\rangle \\ &= \frac{1}{[\sqrt{\det(1-CV)}]^{26}} \end{aligned} \tag{A.18}$$

where $V = V^{11}$, to find that

$$\gamma_3 = \frac{1}{\gamma_1}[\sqrt{\det(1 - CV)}]^{26}\,. \tag{A.19}$$

We note in passing that $V_{nm} = 0$ unless $n \equiv m \pmod 2$, so that V can be decomposed into even and odd blocks V_e, V_o. We use this fact in the following appendix.

In this appendix we have described the relative normalization factors used in the Fock space and split string approaches to string field theory, in the matter sector. One particular advantage of the normalization conventions natural to the split string framework is that rank one projection operators satisfying (2.33) naturally have a unit trace

$$\int \Psi = \mathrm{Tr}\,\hat{\Psi} = 1\,. \tag{A.20}$$

In the following appendix we discuss the connection with the analogous statement in the Fock space description.

B. Normalization of rank one projectors

In [28, 27], a string field $|\Phi_F\rangle$ was constructed using Fock space methods which satisfies the projection equation

$$|\Phi_F\rangle = |\Phi_F\rangle \star_F |\Phi_F\rangle \tag{B.1}$$

defined using the Fock space normalization for the star product $\star_F$ in (A.15). This state is given by

$$|\Phi_F\rangle = [\sqrt{\det(1 - Z)\det(1 + T)}]^{26} \exp\left[-\frac{1}{2}\left(a^\dagger|S|a^\dagger\right)\right]|0\rangle\,, \tag{B.2}$$

where

$$Z = CV\,, \tag{B.3}$$

$$S = CT\,, \tag{B.4}$$

and

$$T = \frac{1}{2Z}\left(1 + Z - \sqrt{(1 + 3Z)(1 - Z)}\right). \tag{B.5}$$

Note that like V, S is only nonzero in the even-even and odd-odd blocks S_e, S_o. The sliver state (B.2) is discussed further in [39, 46].

In [14], it was shown that the state (B.2) is a rank one projector on the space of half string functionals by demonstrating that the dependence of this state on the left and right modes is described by a functional of the form

$$\exp\left(-\frac{1}{2}\,l \cdot M \cdot l - \frac{1}{2}\,r \cdot M \cdot r\right) \tag{B.6}$$

where M is given by

$$M = \frac{1}{2} E_o^{-1} \frac{1 - S_o}{1 + S_o} E_o^{-1}. \tag{B.7}$$

In this expression, $M_{2j+1,2k+1}$ and all other matrices are restricted to the odd-odd block.

We now use the relationship between the Fock space and split string normalization conventions derived above to demonstrate that the state proportional to (B.2) which satisfies the split string projection equation indeed has the normalization (2.34). This serves as a check on our formalism and completes the analytic construction of a rank one projector satisfying $\hat{\Phi} = \hat{\Phi}\hat{\Phi}$ and $\mathrm{Tr}\,\hat{\Phi} = 1$ in the split string formalism.

From the discussion above of the difference between the Fock space and split string normalization of the star product, we see that associated to a state $|\Phi_F\rangle$ satisfying (B.1) there should be a functional

$$\Phi = \frac{1}{\gamma_3} \Phi_F \tag{B.8}$$

which satisfies

$$\Phi = \Phi \star \Phi \tag{B.9}$$

(using the functional integral normalization of the star product (2.17)). The Fock space state (B.2) has the functional description

$$\Phi_F[x] = [\sqrt{\det(1 - Z)\det(1 + T)}]^{26} \frac{k}{[\sqrt{\det 1 + S}]^{26}} \exp\left[-\frac{1}{2} (x|L|x) \right] \tag{B.10}$$

where k is given in (A.2) and

$$L = E^{-1} \left(\frac{1 - S}{1 + S} \right) E^{-1}. \tag{B.11}$$

In [14], we showed that the exponential factor decomposes into a product of Gaussians of the left and right modes of the form (B.6). To show that Φ given by (B.8) is indeed of the form (2.34), it remains to show that the normalization factors match, that is that

$$\left(\det \frac{M}{\pi} \right)^{26/2} = \frac{k}{\gamma_3} [\sqrt{\det(1 - Z)\det(1 + T)}]^{26} \frac{1}{[\sqrt{\det 1 + S}]^{26}}. \tag{B.12}$$

From (A.19) and the definition of $Z = CV$, this reduces to

$$\left(\det \frac{M}{\pi} \right)^{26/2} = k\gamma_1 \frac{[\sqrt{\det(1 + T)}]^{26}}{[\sqrt{\det 1 + S}]^{26}}. \tag{B.13}$$

Breaking S, T into odd and even blocks we have

$$\begin{aligned} \det(1+S) &= \det(1+S_o)\det(1+S_e) \\ \det(1+T) &= \det(1-S_o)\det(1+S_e)\,. \end{aligned} \tag{B.14}$$

From (B.7) we have

$$\det M = \det\left[\frac{E_o^{-2}}{2}\right]\frac{\det(1-S_o)}{\det(1+S_o)}\,. \tag{B.15}$$

It follows that all the determinants of $1 \pm S_{o,e}$ cancel in (B.12), so it remains to verify that the constant factors cancel, which occurs when

$$\det\left(\frac{E_o^{-2}}{2\pi}\right) = k\gamma_1 \tag{B.16}$$

which is equivalent to

$$\prod_{k=0}^{\infty}\left(\frac{2k+1}{2\pi}\right)^{26/2} = \prod_{k=0}^{\infty}\left[\left(\frac{2k}{\pi}\cdot\frac{2k+1}{\pi}\right)\left(\frac{4\pi}{2k}\cdot\frac{2k+1}{\pi}\right)\right]^{26/4}\,. \tag{B.17}$$

which is manifestly true for each k. This completes our check that the state (B.8) is a rank one half-string projection satisfying

$$\Phi = \Phi \star \Phi \tag{B.18}$$

and

$$\int \Phi = 1\,. \tag{B.19}$$

C. Integrals

Here we evaluate an integral used in section 3.1.2.

We wish to evaluate

$$\int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}\frac{e^{-i\frac{\sigma}{2}+i\epsilon(\sigma)\pi/4+in\sigma}}{(2-2\cos 2\sigma)^{1/4}}\,, \qquad n \geq 0\,. \tag{C.1}$$

For $n \geq 0$, we have

$$\int_0^{\pi}\frac{d\sigma}{2\pi}\frac{e^{\pm i(\frac{\sigma}{2}+in\sigma)}}{\sqrt{2\sin(\sigma)}} = \frac{\sqrt{\pm i}}{2\pi}\int_{-1}^{1}dx\frac{x^n}{\sqrt{1-x^2}} = \frac{\sqrt{\pm i}}{2^{n+1}}\binom{n}{n/2}\frac{1+(-1)^n}{2}\,, \tag{C.2}$$

where we changed variables according to $\sigma \to x = \exp(\pm i\sigma)$. Similarly, for $n > 0$ we have

$$\int_0^{\pi}\frac{d\sigma}{2\pi}\frac{e^{\pm i(\frac{\sigma}{2}-in\sigma)}}{\sqrt{2\sin(\sigma)}} = \frac{\sqrt{\mp i}}{2^n}\binom{n-1}{(n-1)/2}\frac{1+(-1)^{n+1}}{2}\,. \tag{C.3}$$

Using these we can easily establish that (for $n \geq 0$)

$$\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} \frac{e^{-i\frac{\sigma}{2}+i\epsilon(\sigma)\pi/4+in\sigma}}{(2-2\cos 2\sigma)^{1/4}} = \delta_{n,0}\,. \tag{C.4}$$

(For $n < 0$ this integral does not vanish). Consequently the integral within the square bracket in (3.44) is identically one and independent of the ϕ_n's. Another way to see this result is to note that

$$\frac{e^{-i\frac{\sigma}{2}+i\epsilon(\sigma)\pi/4}}{(2-2\cos 2\sigma)^{1/4}} = \exp\left(-i\frac{\sigma}{2}+i\frac{\epsilon(\sigma)\pi}{4}+\sum_{m=1}^{\infty}\frac{\cos 2m\sigma}{2m}\right) = \exp\left(\sum_{m=1}^{\infty}\frac{e^{2im\sigma}}{2m}\right) \tag{C.5}$$

so that (C.4) follows immediately.

D. $Q^2 = 0$

In the proof that $Q^2 = 0$, we ignored the fact that the formally vanishing double integral in (3.59) had a nonzero integrand at $\rho = \tau$, which is multiplied by the divergent factor K, so that a finite result might in principle be obtained. If we use the regulated form of Q_f the factor $K^2 e^{i\pi\theta(\tau-\rho)}$, which is odd under $\rho \leftrightarrow \tau$ is replaced by

$$K_x^2 \exp i\left[\frac{\pi}{2}+\frac{1}{2i}\ln\frac{\epsilon+i(\tau-\rho)}{\epsilon-i(\tau-\rho)}\right].$$

In the integral over ρ and τ in (3.59), we can symmetrize in τ and ρ, so that this factor becomes (recall that $K_x \sim 1/\sqrt{1-x}$ and $x = 1-\epsilon$),

$$\frac{i}{\epsilon}\left[\sqrt{\frac{\epsilon+i(\tau-\rho)}{\epsilon-i(\tau-\rho)}}+\sqrt{\frac{\epsilon-i(\tau-\rho)}{\epsilon+i(\tau-\rho)}}\right] = \frac{2i}{\sqrt{(\tau-\rho)^2+\epsilon^2}}\,. \tag{D.1}$$

Therefore, due to the divergent factor of K, the formally vanishing $K^2(e^{i\pi\theta(\tau-\rho)}+e^{i\pi\theta(\rho-\tau)})$ can yield a finite result. However, this does not invalidate the fact that $Q^2\Psi = 0$ for well-behaved functionals Ψ. The essential point is that in (3.59) we encounter

$$\exp(i\phi(\tau)+i\phi(\rho)))\,\Psi\left[\phi(\sigma)+\pi\left(\theta(\tau-\sigma)+\theta(\rho-\sigma)\right)\right],$$

a string field to which two kinks have been added. If Ψ is a nice functional, such as a Fock space state of the form $P(\phi_n)\Psi_0[\phi(\sigma)]$, where $P(\phi_n)$ is a polynomial in the ϕ_n's and Ψ_0 is the vacuum functional

$$\Psi_0[\phi(\sigma)] = Ce^{-i\phi_0/2}\exp\left(-\sum_{n=1}^{\infty}\frac{n}{2}\phi_n^2\right). \tag{D.2}$$

then the shifted vacuum functional is of the form

$$\begin{aligned}
&\exp(i\phi(\tau)+i\phi(\rho))\Psi_0\left[\phi(\sigma)+\pi\left(\theta(\tau-\sigma)+\theta(\rho-\sigma)\right)\right]= \\
&\quad = Ce^{i(3\phi_0-\tau-\rho)/2}\exp\left(i\sqrt{2}\sum_{n=1}^{\infty}\phi_n(\cos(n\tau)+\cos(n\rho))-\right. \\
&\qquad\qquad \left. -\sum_{n=1}^{\infty}\frac{n}{2}\left(\phi_n+\sqrt{2}\frac{\sin(n\tau)+\sin(n\rho)}{n}\right)^2\right) \qquad \text{(D.3)} \\
&\quad = e^{i(4\phi_0-\tau-\rho)/2}\exp\left(i\sqrt{2}\sum_{n=1}^{\infty}\phi_n(e^{in\tau}+e^{in\rho})-\sum_{n=1}^{\infty}\frac{1}{n}(\sin(n\tau)+\sin(n\rho))^2\right)\Psi_0\,. \qquad \text{(D.4)}
\end{aligned}$$

Now we note that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}(\sin(n\tau)+\sin(n\rho))^2= \\
&\quad =\sum_{n=1}^{\infty}\left(\frac{1-\cos(2n\tau)}{2n}+\frac{1-\cos(2n\rho)}{2n}+\frac{2\sin(n\tau)\sin(n\rho)}{n}\right) \qquad \text{(D.5)} \\
&\quad =\sum_{n=1}^{\infty}\frac{1}{n}+\frac{1}{4}\ln(2-2\cos 2\tau)+\frac{1}{4}\ln(2-2\cos 2\rho)+\frac{1}{2}\ln\left(\frac{1-\cos(\tau+\rho)}{1-\cos(\tau-\rho)}\right),
\end{aligned}$$

so that the shifted vacuum functional is

$$\frac{1}{K^2}e^{i(4\phi_0-\tau-\rho)/2}\exp\left(i\sqrt{2}\sum_{n=1}^{\infty}\phi_n(e^{in\tau}+e^{in\rho})\right)\sqrt{\frac{1-\cos(\tau-\rho)}{4|\sin\tau||\sin\rho|(1-\cos(\tau+\rho))}}\Psi_0\,, \qquad \text{(D.6)}$$

where the $1/K^2$ comes from $\exp(-\sum\frac{1}{n})=1/K^2$. This factor of $1/K^2$ cancels the K^2 in Q_f^2, and the rest of the functional is smooth inside the integral in (3.59), which now vanishes due to the antisymmetry of $e^{i\pi\theta(\tau-\rho)}$.

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